HYDROSTATICS

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CHAPTER I

ELEMENTARY PROPERTIES OF FLUIDS. PRESSURE

1. Matter or substance is broadly divided into two classes, (1) *solids* which have definite size and shape under ordinary circumstances, and (2) *fluids* which have no definite shape and which take up the forms of the vessels containing them Fluids again are of two kinds, *liquids* which have definite volume and *gases* which have no definite volumes and have the property of filling up the whole space of the enclosing vessel

Hydrostatics is the branch of mathematics which deals with the equilibrium of a mass of fluid (or of a solid in contact with fluids at rest) under the influence of a given system of forces.

2. **Density** of a substance is defined as its mass per unit volume. For example, the density of water is 62 5 lbs per cu foot, or 1 gramme per cu cm Thus we see that the measure of density depends on the units of mass and volume that are chosen

A body is said to be *homogeneous* (or of uniform density) if equal volumes of it, however small, have equal masses Otherwise the body is said to be *heterogeneous* (or of variable density)

The definition of density given above holds for homogeneous bodies For heterogeneous bodies we define it as follows

Consider an element of the body enclosing a point P, let its mass be m and volume v Then the limit of the ratio $\frac{m}{v}$

when v tends to zero is said to be the *density at the point P*. It follows that if ϱ be the density at the point P and dv an element of volume of the substance surrounding P, its mass will be ϱdv.*

Specific gravity or relative density of a substance is the ratio which its density bears to the density of some standard substance, usually water. It is therefore an absolute number, unlike the ordinary density. If s be the specific gravity of a homogeneous body of volume v, and ϱ denote the density of water (or the standard substance), then the density of the body is denoted by ϱs (in terms of the units of mass and volume chosen), its mass is (ϱsv) and its weight is ($g\varrho sv$) where g denotes the acceleration due to gravity. The value of g is approximately 32·2 ft. sec. units or 981 cm. sec. units.

We have seen before that the measure of density depends upon the units chosen; but the value of the specific gravity is independent of these units. For this reason specific gravity is preferred to density, and is shortly denoted by sp. gr.

Let m, v and m', v' be the masses and the volumes of some quantities of a homogeneous substance and of water respectively. Then their densities are $\frac{m}{v}$ and $\frac{m'}{v'}$.

$$\therefore \text{ the sp. gr. of the substance} = \frac{m}{v} \cdot \frac{m'}{v'} = \frac{m}{m'} \cdot \frac{v'}{v}.$$

From this we get the following particular cases, which are often taken as definitions of specific gravity.

If $v' = v$, the sp. gr. $= \frac{m}{m'}$,

$$= \frac{\text{mass of the substance}}{\text{mass of equal volume of standard substance (water)}}. \quad \ldots(1)$$

* This will be true up to first order of smallness, and so can be used along with infinitesimals of first order only.

If $m' = m$, the sp gr $= \frac{v'}{v}$

$$= \frac{\text{volume of equal mass of standard substance (water)}}{\text{volume of the substance}} \quad \ldots (2)$$

3. Fluid Pressure. Common experience tells us that fluids do not practically offer any resistance to an effort made to separate a part from the rest * If, for example, a blade of knife or a thin plane lamina be moved in its plane across a fluid, very little resistance is experienced. This and similar examples show that in a mass of fluid at rest there does not practically exist, between the different elements of the fluid or between its elements and those of an adjoining substance, a force of the nature of friction (the shearing stress) Therefore, the force between two adjoining portions of the fluid separated by an element of surface or between a portion of the fluid and a solid in contact with it along an element of surface, is wholly normal to this element This force is called the *pressure* exerted by the fluid, or simply fluid-pressure. The existence of this force can be demonstrated by moving a lamina through a fluid at right angles to the plane of the lamina.

But when fluids are in motion the former force is seen to exist, more or less, side by side with the normal pressure This has led to the conception of a purely theoretical substance known as *perfect fluid*, which is defined as one yielding *at once* to the slightest effort made to separate a part from the rest

The following analogous definition can be given to ordinary fluids :

An *ordinary fluid* is a substance which yields to any small effort made to divide it provided the effort be continued long enough

* The case when a fluid is suddenly struck is an exception to this, the force being a very great force acting for a very short time.

The fundamental property of the fluid, which has been discussed above, is often expressed in the form that the pressure of a perfect fluid or of any ordinary fluid at rest is always normal to the surface with which it is in contact. The surface just mentioned may be the surface of separation between the fluid and a solid, between the fluid and another fluid or between two portions of the same fluid.

We have seen that the fluids are of two kinds, liquids and gases. The former may be said to be *incompressible* fluids because their volumes do not change to any appreciable extent, the latter are *compressible* or *elastic* fluids A *perfect liquid* is the name given to an absolutely incompressible fluid, like a perfect fluid, it is an imaginary substance

4. Measure of Fluid Pressure. Pressure at a point. Consider a plane area in contact with a mass of fluid at rest This area is acted upon by the fluid, *i e.* the fluid exerts pressure on it Some force P is therefore necessary to keep the area in position by counterbalancing the action of the fluid P would then measure the fluid *pressure on the whole area* If P be proportional to the area of the plane whatever be its extent, or in other words, if the action of the fluid be uniform over the area, the pressure on the area is said to be *uniform* If, however, the action of the fluid be different on equal portions of the area, the pressure is *varying* In the first case the *pressure at every point* of the plane is defined as $\frac{P}{A}$, A being the measure of the area When the pressure is varying we proceed thus · We take a small portion of the plane, of area a, enclosing a point Q, and let p be the pressure exerted by the fluid on the element Then the limit of the ratio $\frac{p}{a}$ as a tends to

zero, is defined as the *pressure at the point Q* * Thus, if p_1 denotes the pressure at the point Q and dA denotes an element of area enclosing Q, the fluid pressure on the area will be $p_1 dA$ This will be correct up to small quantities of the first order (See Footnote, Art 2, page 2) To be strictly accurate, the pressure on the element should be taken as $(p + \sigma)dA$, where σ is a quantity of the first order, at least, of smallness, since the pressure is not strictly uniform over the element; $p + \sigma$ might be called the mean pressure on it

Fluid pressure on a surface consists therefore of forces acting normally on every element of the surface in the direction *from the fluid to the surface* If the surface be plane the forces form a system of parallel forces, but if it be curved the pressure on its elements will be in different directions So, the pressure on it cannot be said to be uniform in this case, even though its magnitude be the same at all points of the surface

Fluid pressure is sometimes spoken of as fluid thrust.

5. *Pressure at a point in a mass of fluid at rest is the same in every direction*

Take a point A within a mass of fluid at rest and draw an elementary right prism having any right-angled triangle ABC as base, the angle A of the triangle being the right angle The lengths a, b, c of the edges AB, AC, AA' respectively, are small but arbitrary in magnitude, all being taken to be of the first order of smallness The portion of the fluid enclosed within this prism is at rest because the whole mass of the fluid is at rest The forces which act on this element

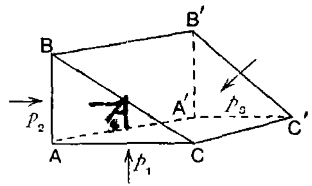


FIG 1

* It is easy to see that this mode of defining the pressure at a point can be applied to the first case also It is therefore taken as the general definition

are (1) the resultant of the impressed forces * which will be proportional to the mass of the element, (2) the pressures on the triangular ends, and (3) the pressures on the rectangular (lateral) faces Since these are in equilibrium, the sum of their resolved parts in any direction will be zero Let us resolve in the direction AB. The force (1) will have $(\varrho abc \cdot k \cos \varphi)$ as its component where ϱ is the density of the fluid element, φ is the angle between AB and the direction of the force and k the constant of variation The forces (2) and the pressure on the face AB' have no component The pressure on the face BC' acts normally to it, and therefore it makes an angle $(180° - \angle ACB)$ with the direction AB. Denoting the pressures at A in the directions AB, AC and perpendicular to BC by p_1, p_2 and p_3, we get, therefore,

$$p_1 \cdot bc - p_3 \cdot c \cdot BC \cos ACB + \varrho abck \cos \varphi = 0, \dots \quad (1)$$

or

$$p_1 bc - p_3 bc + \varrho abck \cos \varphi = 0,$$

or

$$p_3 = p_1 + \varrho ak \cos \varphi.$$

Now, let the elementary prism diminish indefinitely in size so that ultimately $a = 0$, and the prism dwindles into the point A Then we shall get

$$p_3 = p_1$$

Similarly it can be proved, by resolving in the direction of AC, that $p_3 = p_2$

$$\therefore \; p_1 = p_2 = p_3 , \quad \dots \dots \quad (2)$$

i e the pressures at the point A, in the directions AB, AC and perpendicular to BC are the same By altering the ratio a b we can get p_3 to be in any direction that we like, hence the proposition is proved

Note. The actual pressures on faces AC' and BC' are $(p_1 + \sigma_1)bc$ and $(p_3 + \sigma)c \cdot BC$, by Art 4 In the above

* This will be the weight of the elementary prism, if gravity be the only external force acting on the fluid mass

demonstration we have omitted the second part of each as being terms of higher order of smallness than the first part But as in equation (1) the third term of the left-hand member, which is also of higher order, is included, we should not have omitted the above-mentioned terms if we want to be quite rigorous All the same, the final result will not be affected. Taking all the terms, we get, instead of (1),

$$(p_1 + \sigma_1)bc - (p_3 + \sigma)c \quad BC \cos ACB + \varrho abck \cos \varphi = 0,$$

whence $$p_3 = p_1 + \varrho ak \cos \varphi + \sigma_1 - \sigma$$

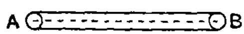
Proceeding to the limit, a, σ_1 and σ vanish ultimately ; therefore we have $p_3 = p_1$, as before

The above theorem means that if we take an elementary plane area through a given point in a mass of fluid at rest, the fluid pressure on it will be the same in magnitude (but not in direction) whatever be the orientation of the plane.

6. Transmissibility of Liquid Pressure. This fundamental principle, known as *Pascal's law*, can be enunciated thus ·

If a pressure be applied to any portion of the surface of a liquid at rest, it is transmitted equally to all parts of the fluid

The property of the liquids stated above is a direct consequence of experimental facts and can be demonstrated by suitable experiments But a theoretical proof can also be given on the basis of the definitions given before



To show that a pressure applied to a point A in the liquid induces an equal pressure at any other point B, also in the liquid Firstly, suppose that the straight line AB lies wholly in the liquid Round AB as axis describe a cylinder of small cross-section α, and consider the equilibrium of the portion of the liquid enclosed within the cylinder It will be at rest since the whole mass of the

liquid is at rest The forces on this cylinder are (1) the pressures pa and $p'a$ on the ends A and B where p, p' denote the pressures at A, B respectively . these forces act along AB or BA , (2) the pressures on the curved surface which are at every point perpendicular to AB , and (3) the resultant external force which has a component, say F, along AB Resolving them along AB, we have

$$pa - p'a + F = 0, \text{ or } p' - p = \frac{F}{a}$$

This shows that if an additional pressure is introduced at A, an equal pressure must automatically come into play at B in order that this equation may be satisfied

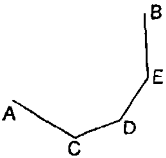


FIG 3

Next suppose that the straight line AB lies partly outside the liquid We can, however, connect A to B by means of a number of connected straight lines AC, CD, DE, EB, as in Fig 3, each of which lies entirely within the liquid Then, by the proposition just proved, the additional pressure at A = the additional pressure at C = that at D = . = that at B

•7. Bramah's Press or Hydraulic Press. The principle given in the foregoing article is utilised in practice in the hydraulic press The object of this machine is to multiply a comparatively small force into a much larger one

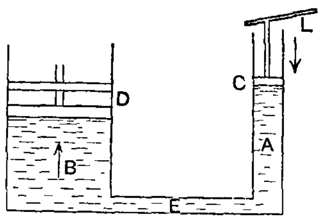


FIG 4

It consists principally of (Fig 4) two vertical cylinders A and B fitted with water-tight pistons C and D. One of these cylinders, B, is much wider than the other The wider cylinder has a tube E near its base leading to the other cylinder The lower parts

of the cylinders and the tube E contain water. The power is applied to the piston C by means of the lever L. This force is conveyed through the water to the piston D of the bigger cylinder.

Suppose the areas of the pistons C and D are c and d respectively; let F be the force applied to the piston C, which induces a pressure p (given by $pc = F$) at every point of water in contact with it. This pressure p is transmitted (by Pascal's law) to every point of water in contact with the piston D. Therefore the total force called into action on D is pd or $\left(\frac{d}{c} \times \text{the applied power}\right)$. Thus, the greater the ratio $d : c$ the greater would be the force that would be exerted upwards on D. An equal pressure acts, at the same time, on the material of which the machine is made. It is not practicable to multiply the effort indefinitely, because some part of the machine may give way before the tremendous force to which it is subjected.

It may be observed, since the volume of water remains unchanged, that $ch = dh'$, where h and h' are the distances through which the pistons C and D may be moved.

$$\therefore pc \cdot h = pd \cdot h',$$

or force on piston $C \times h$ = force on piston $D \times h'$,

or work done on piston C = work done on piston D.

CHAPTER II

THEOREMS RELATING TO PRESSURES OF FLUIDS AT REST UNDER GRAVITY

8. *The pressure of a mass of fluid at rest under gravity alone is the same at every point on the same horizontal plane*

Let A and B be any two points within the fluid in the same horizontal line, and let the line AB be *wholly within* the fluid. About AB as axis describe a cylinder of small cross-section a Let the pressures at the points A and B be denoted by p_A and p_B respectively

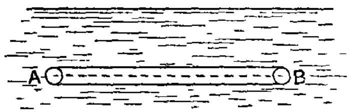


FIG. 5

Consider the equilibrium of the cylinder of fluid and resolve the forces acting on it along AB. The pressures at the two ends are the only forces which have components (or which act) in this direction Hence, $p_A + \sigma$ and $p_B + \sigma'$, denoting mean pressures on the two ends (Art 4),

$$(p_A + \sigma)a = (p_B + \sigma')a, \text{ or in the limit } p_A = p_B, \quad \ldots \quad (1)$$

since σ, σ' vanish in the limit.

The case when the line AB does not wholly lie within the fluid (which is homogeneous) will be discussed in a subsequent article (Art. 13).

Note. The proposition would not hold if there were other forces, besides gravity, acting on the elements of the fluid, for example, in the case of a mass of fluid which is enclosed within a vessel made of flexible material, subject to a number of forces acting at different points of the flexible surface. Hence, *in all succeeding discussions we*

shall assume, unless statement is made to the contrary that the fluid is at rest under gravity only

9. *The pressure at a point A (of fluid) vertically below B, exceeds the pressure at B by the weight of the column of fluid between A and B (the area of the cross-section being taken as unity)*

(i) Let the fluid be homogeneous of density ϱ and let $BA = h$ About AB as axis draw a cylinder of small cross-section a Since the liquid enclosed by this cylinder is in equilibrium, we obtain, by resolving the forces (acting on it) vertically,

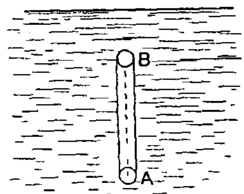


FIG 6

$$(p_A + \sigma)a = (p_B + \sigma')a + g\varrho h a,$$

or $$p_A = p_B + g\varrho h + \sigma' - \sigma.$$

$\therefore$ in the limit,

$$p_A = p_B + g\varrho h = p_B + wh, \quad \dots(2)$$

w denoting the weight of the liquid per unit volume

(ii) If the fluid consists of a number of layers of different fluids (which do not mix) between A and B, the above equation will take the form

$$(p_A + \sigma)a = (p_B + \sigma')a + g\varrho_1 h_1 a + g\varrho_2 h_2 a + \dots,$$

where h_1 is the depth of the fluid of density ϱ_1, h_2 is that of a second fluid of density ϱ_2, ... starting from A to B. Therefore we shall obtain

$$\left.\begin{aligned} p_A &= p_B + g\varrho_1 h_1 + g\varrho_2 h_2 + \dots \\ &= p_B + w_1 h_1 + w_2 h_2 + \dots \end{aligned}\right\} \quad \dots\dots\dots(3)$$

the w's denoting the weights of unit volumes of the corresponding fluids

The result (2) for a homogeneous fluid is sometimes stated as follows

The difference of the pressures at two points varies as the difference of their depths or as the vertical distance between the points. That is,

$$p_A - p_B = g\varrho h, \text{ or } p_A - p_B \propto h.$$

But this statement does not indicate which of the two pressures is the greater, nor does it apply to non-homogeneous fluids

10. If the point B be on the free surface of a homogeneous fluid, and if the value of the pressure at the free surface be zero, the equation (2) gives

$$p_A = g\varrho h = wh, \quad \ldots \quad (4)$$

i e the pressure at a depth h below the free surface (of zero pressure) is equal to the weight of the column of fluid above the point A (the cross-section of the column being taken as unity, as before)

If, however, B be on the free surface of the uppermost fluid, where the pressure is zero, the equation (3) reduces to

$$\left.\begin{aligned} p_A &= g\varrho_1 h_1 + g\varrho_2 h_2 + \ldots \\ &= w_1 h_1 + w_2 h_2 + \ldots \end{aligned}\right\}, \quad \ldots \quad (5)$$

a result which can be included in the above statement

11. In the case of liquids, the results (4) and (5) will hold only if there be no pressure at the free surface But, as usually is the case, the free surface is the surface of separation from the atmosphere, where the pressure is different from zero The pressure at any point of this surface is called the *pressure due to atmosphere* or the atmospheric pressure. It is generally denoted by the height of a column of mercury or of water Thus, the expression that the atmospheric pressure is 30 inches of mercury or 34 feet of water, means that the value of the pressure is equal to the weight of a column of mercury of height 30 inches or of a column of water of height 34 feet, *i e* the pressure $= w \times 30$, where w is the weight of mercury per cu. in, or $= w' \times 34$, where w' is the weight of water per cu ft

As atmosphere is a heterogeneous fluid, its pressure at any point on the surface of the earth is equal in general to the weight of the column of air just above the point But

neither the height of the atmosphere nor the law of its density is known to us, and so we cannot utilise this result to determine the magnitude of the pressure It is determined by an instrument known as the *barometer* (see Chapter IX) 'Water barometer stands at h feet' is equivalent to the statement that the atmospheric pressure is equal to a column of water of height h feet

Let Π denote the atmospheric pressure, then the pressure at depth h of a homogeneous liquid will be given, from (2), by

$$p = \Pi + g\varrho h = \Pi + wh \qquad \ldots \ldots \ldots (6)$$

If $\Pi = g\varrho h_1$ or wh_1, a surface at a height h_1 above the free surface (the surface of separation from atmosphere) of the given homogeneous liquid is called its *effective surface* The pressure at any point of the liquid will then be proportional to its depth below the effective surface, for

$$p = \Pi + wh = w(h_1 + h) = g\varrho(h_1 + h)$$

Comparing this with (4) we see that the effective surface may be regarded as a surface of zero pressure

12. The equation (6) shows that the pressure at a depth h below the free surface of a homogeneous liquid is equal to the sum of the atmospheric pressure and the weight of a column of the liquid of height equal to h

The previous results are established on the assumption that the whole length of the line AB (Art 9) lies within the liquid We shall presently show that (6) or its equivalent statement given above holds even if AB or the vertical line drawn from the point at a depth h to meet the plane of the free surface does not lie wholly within the liquid In the accompanying diagram, the vertical line through A does not meet the free surface of the liquid The free surface is the meeting surface with the atmosphere (or the surface of zero pressure, if it exists). The point C, although on the bounding surface of the liquid, is not on the free

surface. In Fig 7, HK or MN is the free surface ; let the vertical through A meet its plane at B'. Then we shall have, as in (6),

$$p_A = \Pi + w \cdot AB'$$
$$= \Pi + g\rho \cdot AB'.$$

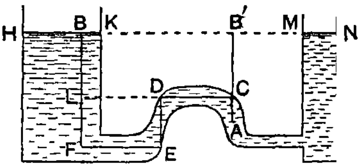


FIG 7.

To show this, connect A to B (a point on HK) by means of vertical and horizontal lines, AC, CD, DE, EF and FB, so that each line lies wholly in the liquid. From Arts. 8 and 9,

$$\begin{aligned} p_A &= p_C + w \cdot AC = p_D + w \cdot AC \\ &= p_E - w \cdot ED + w \cdot AC \\ &= p_F + w \cdot (AC - ED) \\ &= p_B + w(AC - ED + BF) \\ &= \Pi + w \cdot AB', \end{aligned}$$

since $p_B = \Pi$ and $AC - ED + BF = AB'$.

13. We are now in a position to extend the proof of Art. 8 to the case when the whole of the horizontal line joining the two points does not lie in the liquid. Let C and L (Fig. 7) be two points in the liquid in the same horizontal level. Join C to L by means of horizontal and vertical lines CD, DE, EF and FL Then, as in the last article,

$$\begin{aligned} p_C = p_D &= p_E - w \cdot ED = p_F - w \cdot ED \\ &= p_L + w \cdot LF - w \cdot ED = p_L, \end{aligned}$$

since $LF = DE$.

14. *If two fluids which do not mix are in equilibrium in contact with each other, their surface of separation is a horizontal plane.*

The two fluids may be a gas and a liquid, or both of them may be liquids ; but rarely they are two gases, because the

gases possess the power of mixing with each other with comparative ease—a process known as diffusion

Let A and B be any two points on the common surface. It is to be proved that AB is horizontal Take two points C, D in the lower liquid in the same horizontal level, C vertically below A and D below B If $AC = BD$, then AB would be parallel to CD and therefore horizontal

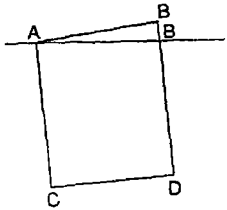


FIG 8

If possible let $AC > BD$, so that AB is not horizontal From A draw the horizontal line AB', meeting DB produced in B'; then B' is in the upper liquid Then, by Art 8,

$$p_A = p_{B'} \quad \text{and} \quad p_C = p_D.$$

$$\therefore\ p_C - p_A = p_D - p_{B'}$$

$\therefore$ by Art. 9, $\quad g\varrho\ AC = g\varrho\,.\,DB + g\varrho'\ BB'$,

where ϱ, ϱ' are the densities of the lower and the upper liquids

$$\therefore\ \varrho(AC - BD) - \varrho' BB' = 0,$$

or because $AC = B'D$, $\qquad (\varrho - \varrho')\,.\,BB' = 0$,

which is impossible, since neither $(\varrho - \varrho')$ nor BB' is zero. Hence AB must be horizontal It follows therefore that the surface of separation is a horizontal plane.

15. If a liquid is at rest in contact with atmosphere, its free surface is a horizontal plane, so would be its effective surface (or surface of zero pressure)

For let A and B be two points (Fig 8) on the effective surface, then $p_A = p_B = 0$ By Art 8, AB must be horizontal Otherwise thus

$p_C = g\varrho\ AC$ and $P_D = g\varrho\ BD$; since $p_C = p_D$, $AC = BD$. Therefore AB is parallel to CD and thus horizontal.

It is characteristic of fluids that the lower of the two fluids in contact shall be denser than the other, *i e.* the density of the lower fluid > the density of the upper.

Another characteristic of liquids is expressed by saying that a liquid always finds its own level It means that when two vessels containing the *same* liquid are put into communication with each other, the liquids in the two vessels get themselves adjusted till the levels form parts of the same horizontal plane In other words, the two different parts of the surface of separation of the liquid from the atmosphere are in one plane, as in Fig 7, HK and MN are in the same level This follows from the previous theorem, by taking A, C and B, D (Fig 8) in the two vessels respectively Then $p_C = p_D$ by Art 13, although the line CD does not lie wholly in the liquid, and the proof follows exactly as in Art 14 This is true only when the liquids in the two vessels are of the same kind

16. Illustrative Examples. *Ex* 1 A rectangular cistern whose dimensions are 2 ft, 3 ft and 2 ft is filled with equal volumes of water and oil (sp gr 0 7) Find (1) the pressure at any point of the base, (2) the pressure on the whole base, the atmospheric pressure being equal to 30 inches of mercury (sp gr 13 6)

Since the depth of the cistern is 2 ft, the depths of oil and water are 1 foot each Weight of unit volume of oil is 0 7 times, and that of mercury 13 6 times the weight of unit volume of water Also 1 cu ft of water weighs 62 5 lb.

$\therefore$ atmospheric pressure = wt of a column of mercury 30 inches high (of unit cross-section)

$$= 13 \cdot 6 \times 62 \cdot 5 \times \tfrac{30}{12} \text{ lb.}$$

$$= 2125 \text{ lbs weight.}$$

This is the value of pressure per sq foot, since one foot is taken as unit of length.

Pressure at any point of the base is, by Arts 9 and 11,

$$=2125 \text{ lb} + w_1h_1 + w_2h_2$$
$$=2125 \text{ lb.} + (62{\cdot}5 \times 1) \text{ lb} + (0{\cdot}7 \times 62{\cdot}5 \times 1) \text{ lb}$$
$$=2125 \text{ lb} + 62{\cdot}5 \text{ lb} + 43{\cdot}75 \text{ lb} = 2231{\cdot}25 \text{ lbs weight.}$$

Since the base is horizontal, the pressure at every point is the same and equal to the above value Therefore the pressure on the whole base

$$= \text{area of the base} \times \text{pressure at a point}$$
$$= (2 \times 3) \times 2231{\cdot}25 \text{ lb}$$
$$= 13387{\cdot}5 \text{ lbs weight}$$

Ex 2 A closed hollow vessel is in the shape of a cone with the base which is horizontal and downwards, water is poured into it through a small hole at the vertex till the vessel is full Find the pressure at any point of the base, and show that the total pressure on the base is three times the weight of the water contained, if the atmospheric pressure be neglected

Let P be any point on the base, draw PQ vertical meeting the cone at Q It should be noted that Q is not on the free surface of water in the vessel, since there is water at a higher level than this, and since it does not lie on the common surface of the atmosphere and water The free surface is at A, although it is of very small area, being in size equal to that of the aperture at the point. Connect P to A by means of horizontals and verticals such as PQ, QE, EA or PD, DA Then, neglecting atmospheric pressure, the pressure at P

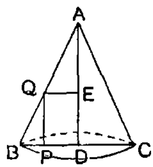


FIG 9

$$= w(PQ + EA) \text{ or } w \cdot DA,$$

where w denotes the weight of unit volume of water Since the base is horizontal, the pressure at every

point is the same. Therefore the total pressure on the base

$$= w \,.\, DA \times \pi BD^2 = w \,.\, \pi \,.\, BD^2 \,.\, DA$$

$= 3$ times the weight of water occupying the cone.

Note. If the cone were completely closed and just full of water, the free surface would still be at A, since there is no liquid at a higher level than this, and the pressure here would actually be zero.

Ex. 3 A cylindrical vessel is fitted with a heavy top which rests on the rim of the vessel. There is a long vertical tube attached to the cylinder just near its top Liquid of density ϱ is poured in through this tube till it fills the vessel and a part of the tube Find the height of the liquid in the tube so that the top be just lifted

Let $AB = h$ be the length of the vertical tube occupied by the liquid when the top BC is just lifted ; let the weight of the lid be W. The pressure of the liquid at every point of the lid is upwards (Art 4) and $= \Pi + g\varrho h$, because the free surface (at A) is at a height h above its plane, Π is the atmospheric pressure.

$\therefore$ the total pressure on the lid $= (g\varrho h + \Pi)A$, where A denotes its area When this pressure just balances the weight W of the lid and the air pressure ΠA on its top, the lid is on the point of being lifted. Therefore the height h is given by $g\varrho hA = W$.

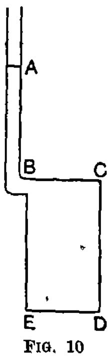


FIG. 10

Ex 4. A U-tube of uniform cross-section contains some mercury. Water is then poured into one limb and occupies a length of 6 inches Find the distance through which the mercury level in the other limb is raised Sp. gr of mercury is 13·6.

Let A, B denote the initial levels of mercury ; they are in the same horizontal plane. Let DA' be the length of

water poured, and let B' be the mercury level in the other limb. It is obvious that the level A is depressed to A' and the level B raised to B', so that $AA' = BB' = x$ inches (say). Take two points in the *lower* liquid and at the same level. The pressures at these points are equal by Art. 8 Take the two points to be A' and C. Then $p_{A'} = \Pi + w \cdot DA'$ and $p_C = \Pi + 13{\cdot}6w \cdot B'C$, where Π represents the atmospheric pressure. Hence equating we get

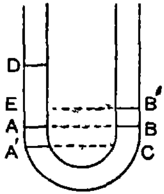


FIG 11.

$$DA' = 13{\cdot}6 \cdot B'C, \text{ or } 6 = 13{\cdot}6 \times 2x.$$

$$\therefore\ x = \frac{3}{13{\cdot}6} = \frac{15}{68}.$$

Note. In equating pressures at two points on the same horizontal level care must be taken in selecting the two points in the same liquid, for, in the above example the line joining the two points does not lie wholly in the liquid as in Art 8, nor is the fluid homogeneous (*i.e* of the same density from D to B') as in Art. 13. For example, the pressure at $B' = \Pi$, whilst the pressure at E (on the same horizontal level) $= \Pi + w \cdot DE$, and these are not equal.

This difficulty can also be evaded by taking the values of the two pressures at the same point, one due to each level, viz D and B' The most convenient point for this purpose is the lowest point of the tube. Next example shows how this can be done)

Ex. 5. A circular tube of uniform thin bore is half filled with equal volumes of three liquids (which do not mix) of sp. gr. 3, 4 and 6, and is kept with its plane vertical Find the inclination, to the vertical, of the diameter joining the two free surfaces of the fluid

Let AD be the diameter in question, the arcs AB, BC, CD being the lengths occupied by the three liquids. Then

$\angle AOB = \angle BOC = \angle COD = 60°$ It is clear that BC is the heaviest liquid (since it is lowest) and CD the lightest (since it occupies the highest position) Let the angle $DOL = \theta$, EOL being the vertical diameter Draw DL, AK, CH, BF perpendiculars to EOL Then the pressure (p_E) at the lowest point E is due to the liquid ABE as also due to the liquid ECD. Therefore these two values for p_E must be equal Let w be the weight of unit volume of water

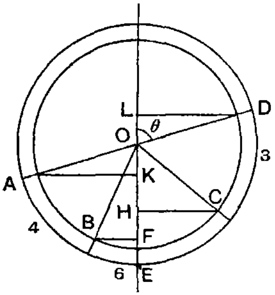


Fig 12

$$p_E = \text{pressure due to } ABE$$
$$= 6w \cdot FE + 4w \cdot FK$$

Also, $$p_E = \text{pressure due to } ECD$$
$$= 6w \cdot EH + 3w \cdot HL.$$

Hence equating and removing the common factor w,

$$6FE + 4FK = 6EH + 3HL, \text{ or } 4FK = 6FH + 3HL$$

Substituting their values in terms of θ, we get, if a = radius of the circle,

$$4a[\cos(\theta - 60°) - \cos\theta]$$
$$= 6a[\cos(\theta - 60°) - \cos(120° - \theta)]$$
$$+ 3a[\cos\theta + \cos(120° - \theta)],$$

whence we shall have $\tan\theta = \dfrac{19}{\sqrt{3}}$

EXAMPLES. 1.

[*N.B* —Atmospheric pressure should be neglected in solving the following examples unless indications are to the contrary]

1 Water is to be supplied to a town from a reservoir, the flow of water being regulated by gravity only If the pipes can bear a maximum pressure of 150 lbs per sq. in , find how high can the level of water in the reservoir be above the average level of the town. [1 cu. ft of water weighs 62 5 lb]

A right circular cone which is placed with its axis vertical and vertex downwards contains glycerine (sp. gr. 1·25) to a depth of 8 in Seven times as much water in volume is then added to the top of the glycerine If the two liquids do not mix, find the pressure in lbs (per sq ft) at the lowest point of the cone, the atmospheric pressure being due to a column of water of height 32 ft

3. If a triangle be immersed in a homogeneous liquid, prove that the sum of the pressures at the vertices is equal to three times the pressure at the centroid of the triangle.

4. The pressures at two points A and B in a homogeneous liquid are p and p' Express the pressure at the point M which divides AB in the ratio m n, in terms of p, p'.

5. If a parallelogram be immersed in any manner in a homogeneous liquid, prove that the sum of the pressures at the extremities of each diagonal is the same

6 If there are n heavy fluids arranged in strata of equal thickness d, whose densities are ϱ, 2ϱ, 3ϱ, . $n\varrho$, beginning from the uppermost, find the pressure at any point of the base of the lowest stratum, hence prove that the pressure at any point of a fluid whose density varies as the depth is proportional to the square of the depth

7 If vessels whose bases are horizontal and equal in area contain the same liquid, prove that the total pressure on the base of each vessel is proportional to the depth of the liquid in it and does not depend on the form of the sides of the vessels.

8. A long tube of thin uniform bore is in the form of three sides of a rectangle, the middle side being horizontal and the others vertical, the length of the horizontal portion is 4 in Mercury (sp gr 13 6) is poured in from one end and water from the other If mercury occupies a length 9 in of the tube and water 17 in, find the position of the surface of separation of the two liquids

9. In the above example, what must be the length of the tube occupied by mercury so that the common surface be at the middle point of the horizontal tube, water occupying a length 17 in ? How much water must now be added on the top of the mercury so that the common surface may shift to a corner ?

10 Two cylindrical vessels whose lengths are a and b are placed on the same horizontal plane, and are connected at the bases by a short tube of thin bore If two liquids which do not

mix and whose densities are ϱ and ϱ' fill the vessels completely, determine the lengths of the vessels occupied by each liquid. [Neglect the volume of the connecting tube, and take $\varrho > \varrho'$ and $b > a$.]

11. A thin tube, whose two straight branches are inclined at an angle a, contains two liquids which do not mix. When one of the branches is held vertically, the liquids meet at the angle of the tube; if now the tube be held with the other branch vertical, prove that the length of the liquid that remains in this branch is $(\cos a)$ times the length of the liquid that was previously in it.

12. A tube of the form of a complete circle, fixed in a vertical plane, contains equal lengths of four liquids which do not mix and whose sp. gr. are as $1 : 2 : 4 : 3$, filling the tube in this order. Find the inclinations to the vertical of the two diameters joining the points of division of the liquids.

13. A thin uniform tube, in the form of a circle whose plane is vertical, contains equal lengths of four liquids whose densities are as $3 : 4 : 6 : 5$, which fill half the tube. The liquids do not mix with one another, and they remain in the tube in the given order of their densities. Find the inclination, in the position of equilibrium, of the diameter joining the free ends, to the vertical.

14. A tube of fine uniform bore and length $(a+b)$, whose plane ends are perpendicular to its length, is bent into the form of an angle. The longer arm whose length is a is kept vertical and the other arm is closed by a loose-fitting cap of weight W, at the upper end. The tube contains liquid which fills the oblique arm and part of the vertical one. It is observed that the cap is just being lifted when the lengths of the liquid in the two arms are equal. Show that $W = bw(\sec a - 1)$, where a is the angle between the arms and w is the weight of unit length of the liquid.

15. A fine tube of uniform bore, bent into the form of an arc of a parabola bounded by a double ordinate equal to twice the latus rectum, is held with its axis vertical and vertex downwards. It is filled with three liquids (which do not mix) whose densities taken in order are as $10 : 13 : 2$. If the line joining the points of separation of the liquids passes through the focus of the parabola, show that its inclination to the vertical is 60°.

16. A closed tube in the form of an ellipse with its major axis vertical, is filled with three liquids of densities ϱ_1, ϱ_2, ϱ_3. P_1 is the point of separation of liquids of densities ϱ_2, ϱ_3, P_2 that of liquids of densities ϱ_3, ϱ_1; and P_3 that of liquids ϱ_1 and ϱ_2. If the distances of P_1, P_2, P_3 from the same focus be r_1, r_2, r_3 respectively, prove that

$$r_1(\varrho_2-\varrho_3)+r_2(\varrho_3-\varrho_1)+r_3(\varrho_1-\varrho_2)=0.$$

17. A large spherical shell of small uniform thickness and weight W, is just full of water. A small circular part of the shell has been cut out some distance from the top of the sphere and hinged at the top of the aperture, fitting the latter closely so as to be watertight. If W' be the weight of water filling the shell, prove that no water escapes through the aperture if the angular distance of the aperture from the top of the shell be less than $\cos^{-1}\dfrac{3W'}{W+3W'}$.

CHAPTER III

PRESSURE ON PLANE AREAS. CENTRE OF PRESSURE

17. We have seen that fluids exert normal forces (called fluid-pressure) on every element of the surfaces with which they are in contact In the case of a plane surface the pressures on all elements form a system of parallel forces, and hence their resultant can be obtained by the rule of compounding parallel forces The magnitude of this resultant will be equal to the sum of the pressures on all the elements, and its line of action can be determined by the rule given in Statics The point where this line meets the plane is called its *centre of pressure* As the magnitudes of the pressures on the various elements change with the position of the plane area, it is clear that the position of the centre of pressure depends, in general, on that of the area relative to the fluid (or fluids) with which it is in contact

18. *To determine the magnitude of the total or resultant pressure on a plane area in contact with fluid* (i) We shall first consider the case of a homogeneous fluid, the pressure on whose free surface is zero. Let the measure of the area be A Divide the whole area into small elements such as a at a point P, let the depth of P below the free surface be z, and w be the weight of unit volume of the fluid Then

$$\text{the pressure on the element} = p_P \cdot a = wza,$$

from Art 10, (4)

$\therefore$ summing up for all the elements, we get total pressure

$$= \Sigma wza = w\Sigma za = wzA = g\rho zA, \ldots \quad \ldots (1)$$

where $\bar{z}$ denotes the depth of the centre of gravity of the area [since from Statics the coordinates of the c g are given by formulae of the type $\bar{z}=\frac{\Sigma za}{\Sigma a}=\frac{\Sigma za}{A}$]. Thus we get the useful theorem·

The resultant pressure of a homogeneous fluid on a plane area is equal to the product of the depth of the c g of the area below the surface (where pressure is zero), the measure of the area and the weight of unit volume of the fluid

(ii) We shall next take the effect of the pressure of the atmosphere, *i e* we shall take the pressure at any point of the free surface of the fluid to be Π, the atmospheric pressure In this case,

$$\text{the pressure on the element} = p_r \cdot a = (\Pi + wz) \cdot a$$

$$= \Pi a + wza$$

∴ summing, we get the total pressure

$$= \Sigma \Pi a + \Sigma wza = (\Pi + w\bar{z})A \quad \ldots \quad . \quad (2)$$

If $\Pi = wh_1$ where h_1 is the height of the effective surface above the free surface (Art 11), the total pressure is given by

$$w(h_1+\bar{z})A, \text{ or } w\bar{z}_1 A, \quad \ldots \ldots \ldots \quad (3)$$

where $\bar{z}_1$ is the depth of the c g of the area below the effective surface Since the pressure at the effective surface is zero, it is evident that this result can be included in the theorem given above

19. Let us next suppose that the fluid consists of layers of different liquids of densities $\rho_1, \rho_2, \ldots \rho_n$ and of depths $h_1, h_2, \ldots h_n$ respectively beginning from the top, and that the area is entirely in contact with the lowermost liquid All the surfaces of separation are horizontal planes, the uppermost being the free surface of the whole mass of the fluid, let the pressure at this surface be zero and the

depth of the point P of the area below the lowest surface of separation be z. Then the pressure at P

$$=g\rho_1 h_1+g\rho_2 h_2+\ldots+g\rho_{n-1}h_{n-1}+g\rho_n z,$$

by Art 10 (5). Therefore the total pressure on the area

$$=\Sigma a\left(g\rho_1 h_1+g\rho_2 h_2+\ldots+g\rho_{n-1}h_{n-1}+g\rho_n z\right)$$

$$=\left(g\rho_1 h_1+g\rho_2 h_2+\ldots+g\rho_{n-1}h_{n-1}+g\rho_n \bar{z}\right)A, \quad \ldots\ldots (4)$$

where $\bar{z}$ denotes the depth of the c g of the area below the lowest surface of separation. If, however, the atmospheric pressure, Π, is taken into account, it is given by

$$\left(\Pi+g\rho_1 h_1+g\rho_2 h_2+\ldots+g\rho_{n-1}h_{n-1}+g\rho_n \bar{z}\right)A, \ldots\ldots (5)$$

as can be proved easily.

If the plane area be in contact with more than one fluid, we divide it into several parts, each of which is in contact with one fluid only. The pressures on each part will be given by the help of the above formulae and their sum is the total pressure required.

The above formulae will be of use when we know the magnitude of the area and the position of its c g Otherwise we have to determine them before we can use the formulae. The following general method enables us to calculate directly the total pressure on the area , the cases where the c g or the measure of the area is not readily obtained, can be better treated by the process indicated in the next article.

20. Take the intersection of the plane of the area with the free surface of the liquid if it is homogeneous, or with that of the topmost liquid if it consists of a number of layers of different liquids, as the axis of x and any convenient perpendicular line in the plane as the axis of y It will simplify the calculation if the axis of y be an axis of symmetry of the area. Divide the area into thin parallel strips by lines drawn parallel to Ox, one of them being indicated in the figure as $PQQ'P'$. Let xOL be the free

surface, $ON = y$, $ON' = y + dy$ Since the strip is horizontal and of infinitesimal breadth, the mean value of the pressure over this area will be $p_N + \sigma$ where σ is an infinitesimal. Therefore the pressure over the strip

$$= (p_N + \sigma) \quad PQdy = p_N \quad PQdy, \quad \ldots\ldots\ldots\ldots \quad (6)$$

up to first order, the higher order terms being neglected. In subsequent articles we shall take this value of the

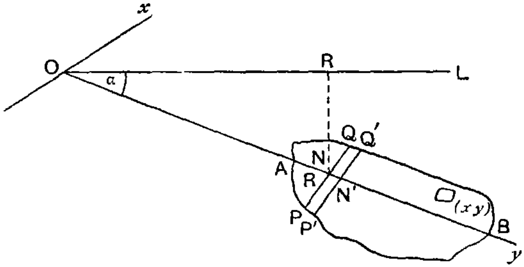


FIG 13

pressure over such elementary strips, the rule to be followed is that the product of the area of the element and the pressure at any convenient point in it gives the pressure on the element up to first order *

Thus the pressure on the element

$$= g\varrho \quad NR \,.\, PQdy = g\varrho y \sin \alpha \quad PQdy,$$

if the liquid is homogeneous and the pressure at the free surface is zero

$$\therefore \text{ the pressure on the whole area} = \int_A^B g\varrho y \sin \alpha PQdy,$$

the limits of integration being chosen to cover the whole area, thus the total pressure

$$= g\varrho \sin \alpha \int_A^B yPQdy = g\varrho \sin \alpha \int_A^B ydA \quad \ldots\ldots\ldots (7)$$

* Those who do not want such rigour may at once take the pressure on the element to be $p_N \,.\, PQdy$, the argument being that the pressure at any point of the strip is the same.

This would give the result as $g\varrho \sin\alpha \cdot \bar{y}S$, as before, where S is the value of the area, since the ordinate $\bar{y}$ of the c g is, in Statics, given by

$$\bar{y} \cdot S = \int y\,dA.$$

The general formula for the total pressure is, however,

$$\int_A^B p_x PQ\,dy, \text{ or } \int_A^B p_x \cdot dA \quad \ldots \ldots \qquad \ldots (8)$$

If the atmospheric pressure is to be taken into consideration, p_x must be replaced by $\Pi + g\varrho y \sin\alpha$, and if the liquid is not homogeneous the value of p_x that should be used is that given in Art 10 (5) The value of PQ can be determined from the shape of the boundary of the area

21. Examples. *Ex* 1 Find the pressure on a triangular area, the depths of whose vertices are h_1, h_2 and h_3, the liquid being homogeneous

The depth of the c g of the triangle is given by

$$\tfrac{1}{3}(h_1 + h_2 + h_3)$$

∴ total pressure, by (1), $\tfrac{1}{3}w(h_1 + h_2 + h_3)A$

Ex 2 A rectangular area is in contact with two liquids of densities ϱ_1 and ϱ_2, the depth of the lighter liquid (of density ϱ_1) is h If the area has its side a on the surface of the upper liquid and the side b vertical, calculate the pressure on the rectangle

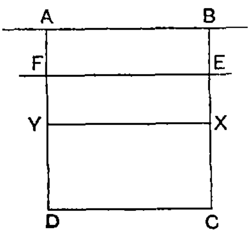


FIG 14

Let EF denote the surface of separation. Then $BE = h$ Pressure on rect AE, by (1),

$$= g\varrho_1 \frac{h}{2} \cdot ah = \frac{1}{2} g\varrho_1 ah^2,$$

and that on rect $ED = g\varrho_1 h \cdot a(b-h) + g\varrho_2 \dfrac{b-h}{2} \cdot a(b-h)$,

by (4). Hence the total pressure on the whole rectangle is their sum, which is, after simplification,

$$=\tfrac{1}{2}g\varrho_1 ah(2b-h)+\tfrac{1}{2}g\varrho_2 a(b-h)^2$$

Ex 3 In the last example, if $\varrho_2=2\varrho_1$ and $h=\frac{1}{4}b$, show how to divide the area by a horizontal line so that the pressures on the two parts may be equal

Let (Fig 14) XY be the required line; let $BX=x$. Then pressure on rect AX is obtained by putting x for b, $\frac{1}{4}b$ for h and $\varrho_2=2\varrho_1$ in the previous result Thus it is

$$\tfrac{1}{32}g\varrho_1 a[b^2-8bx+32x^2]$$

And the pressure on rect XD, by (4),

$$=\tfrac{1}{4}g\varrho_1 a[3b^2+bx-4x^2]$$

Since the pressures are equal, we have, by equating,

$$b^2-8bx+32x^2=24b^2+8bx-32x^2,$$

or $$64x^2-16bx-23b^2=0,$$

$$\therefore\ x=\tfrac{1}{8}(2\sqrt{6}+1)b,$$ neglecting the negative root

Note. Instead of calculating the pressure on the rect XD, we might have calculated that of the whole rect, which is $\frac{25}{32}g\varrho_1 ab^2$ from last example Then the pressure on rect $AX=\frac{1}{2}$ pressure on rect AC, whence the value of x can be determined

Ex 4 Find the pressure on a quadrant of a circle placed with one radius horizontal and at a slant depth b below the surface, the radius being a and the inclination of the plane to the free surface being α

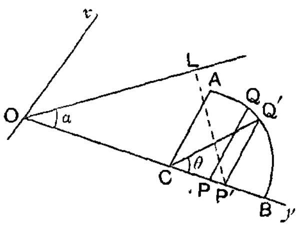


FIG 15

We proceed as in Art 20, and take the other radius CB as the y-axis Let $PQQ'P'$ be the element, and let

$$\angle P'CQ'=\theta,\ \text{and}\ \angle Q'CQ=d\theta.$$

Then the area of element $= P'Q' \,.\, PP' = a \sin\theta \,.\, ad\theta \,.\, \sin\theta$, since PP' = projection of QQ'; and pressure at P'

$$= g\varrho \,.\, OP' \sin\alpha = g\varrho\,(b + a\cos\theta)\sin\alpha$$

$\therefore$ pressure on the element

$$= g\varrho a^2 \sin^2\theta\,(b + a\cos\theta)\sin\alpha d\theta$$

$\therefore$ the total pressure

$$= g\varrho a^2 \sin\alpha \int_0^{\pi/2} (b + a\cos\theta)\sin^2\theta d\theta$$

$$= \tfrac{1}{12} g\varrho a^2 \sin\alpha\,(3\pi b + 4a).$$

The same result would be obtained by formula (1), the c.g. of the area being at a distance $\frac{4a}{3\pi}$ from CA.

Ex. 5. An ellipse is placed with its minor axis on the surface of water and its plane vertical; a circle is described on the minor axis as diameter. Find the pressure of water on the portion of the area enclosed between the ellipse and the circle

First method. Pressure on the area required = pressure on semi-ellipse − pressure on the semi-circle

$$= w\frac{\pi ab}{2} \cdot \frac{4a}{3\pi} - w \,.\, \frac{\pi b^2}{2} \cdot \frac{4b}{3\pi}, \text{ from Art. 18 (1)},$$

$$= \tfrac{2}{3} wb\,(a^2 - b^2),$$

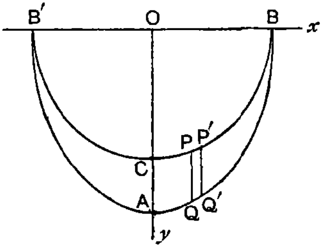


FIG. 16.

since the c g.'s of the semi-ellipse and the semi-circle are at distances $\frac{4a}{3\pi}$, $\frac{4b}{3\pi}$ from the centre O (on the axis OA).

A different method is given below.

Second method Divide the area into thin vertical strips like $PQQ'P'$ (as shown in the figure). Let the distance of PQ from Oy be x and the thickness of the strip be dx.

Pressure on the strip $= w \times$ area $\times$ depth of its c g.

$$= w \times (y_Q - y_P)\,dx \times \tfrac{1}{2}(y_P + y_Q)$$

$$= \frac{w}{2}(y_Q^2 - y_P^2)\,dx$$

Now $y_Q^2 = a^2\left(1 - \dfrac{x^2}{b^2}\right)$ and $y_P^2 = b^2 - x^2$, since Q is on the ellipse and P on the circle.

$$\therefore\ y_Q^2 - y_P^2 = (a^2 - b^2)\left(1 - \frac{x^2}{b^2}\right).$$

$$\therefore\ \text{the total pressure} = \int_{-b}^{+b} \frac{w}{2}(a^2 - b^2)\left(1 - \frac{x^2}{b^2}\right)dx$$

$$= \tfrac{2}{3}wb(a^2 - b^2).$$

The division of the area (in the above case) into horizontal strips is not so convenient, since we shall get two different expressions for the area of the strip according as it is above or below the point C.

EXAMPLES. 2.

1. The portion of a sloping embankment, which is in contact with water of a reservoir, is 200 ft. by 25 ft and the depth of water in the reservoir is 16 ft. Find the resultant fluid thrust on the embankment. [1 cu. ft of water weighs 62 5 lbs.]

2. A cube, each edge of which is 6 inches in length, lies at the bottom of a tank (full of water) 5 feet deep. Calculate in lbs. the resultant fluid pressure on the upper horizontal face and on one of the vertical faces of the cube, assuming the height of water barometer to be 34 ft

3. A rectangular vessel is full of water; compare the fluid pressures on the lower and the upper halves of a vertical side of the vessel

4. One-third of a rectangular vessel is filled with mercury (sp gr 13 6) and the remainder with salt water (sp. gr. 1·04). Determine the fluid thrust on a vertical side.

5. An equilateral triangular prism is completely immersed in water with one lateral edge in the surface and one lateral face vertical Show that the fluid thrusts on the three lateral faces are as 1 2 : 3

6. The resultant fluid pressure on a vertical circle of radius a is equal to twice the weight of a sphere (of the same liquid) of radius a. The circle is now lowered through a depth equal to $2a$ Find the fluid thrust in the new position.

7. The corner A of the triangle ABC, whose plane is vertical, is fixed at a depth h below the surface of a homogeneous liquid. The triangle is turned round this point in its plane and is always completely immersed Determine the positions of the triangle when the fluid thrust on it is maximum and minimum Compare the values of these thrusts

8. A closed conical vessel is just filled with water and the whole is suspended from a point of the rim of the base Neglecting the weight of the vessel, calculate the fluid thrust on the base in the position of equilibrium, given that the radius of the base is r and the semi-vertical angle of the cone is α.

9. A hollow weightless hemisphere (with base) is filled with a liquid and suspended freely from a point in the rim of its base. Determine the resultant pressure on the base [The distance of the c g of a hemispherical volume from the centre is $\frac{3}{8}$ of the radius]

10. A triangular area is immersed in a homogeneous fluid with the vertex in the surface and the base horizontal Give constructions for drawing a horizontal line which would divide the area into two portions the fluid thrusts on which are equal

11. The lighter of two liquids, of density ρ, rests on the other (density σ) to a depth b A square of side a is immersed with its plane vertical and one side in the surface of the upper liquid Prove that the fluid thrusts on the two portions of the square in contact with the two liquids will be equal, provided that $3b > 2a$ and that $\sigma(a-b)^2 = \rho b(3b-2a)$

12. The lighter of two fluids which do not mix and whose sp gr are as 2 : 3, rests on the heavier to a depth of 4 inches A square is immersed with one side in the upper surface, its plane being vertical Show that the length of a side of the square, in order that the thrusts on the two portions in the two liquids may be equal, will be 5·55 inches approximately.

13. A rectangle $ABCD$ is immersed in a homogeneous liquid with the side AB in the surface and AC vertical. Show how to divide the area by a straight line through A so that the fluid thrusts on the two portions may be equal

14. A hollow regular tetrahedron is filled with liquid of density ρ and held so that two opposite edges (length a) are horizontal. Find the thrusts on its faces

15. A circular area is immersed in a homogeneous liquid touching the surface at the point A Draw a chord BC of the circle perpendicular to the diameter AD, so that the pressure on the triangle ABC may be a maximum, also find the ratio of fluid thrusts, in this case, on the triangle and the circle.

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16. A hollow closed pyramid is of weight W and height h, its base is a square of side a, and the slant edges are all equal. It is kept with the base horizontal and is partially filled with water through a hole at the vertex What will be the depth of water inside the pyramid so that the sum of the vertical components of the fluid thrusts on the four lateral faces is equal to W?

17. A layer of water rests upon a liquid of sp gr 1 5 with which it does not mix A triangle of altitude h is immersed vertically in the two liquids so that the base of the triangle is in the surface of water If the thrusts on the two parts of the triangle, which are in contact with the two liquids, be equal, show that the depth x of the water satisfies the equation

$$7x^3 - 9hx^2 - 3h^2x + 3h^3 = 0$$

18. A rectangular vessel contains n heavy liquids, arranged in strata of equal thickness, whose densities are $\rho, 2\rho, 3\rho, \ldots n\rho$ beginning from the top Find the fluid thrust on one of its vertical sides, also those on the two halves into which this side is divided by a diagonal

19. A rectangle is immersed in a homogeneous liquid with one of its edges on the surface Show how to divide the area, by horizontal lines, into n parts, the thrusts on which are equal.

20. A semi-circle is immersed vertically in a liquid with the diameter in the surface, show how to divide it into n sectors such that the pressure on each may be the same

21 A rectangular area $ABCD$ is subject to fluid pressure which is given by the expression $\dfrac{1}{a}\varphi'(x)$ at a distance x from AB If $AB=a$, $BC=b$, prove that the total pressure on the area is $\varphi(b)-\varphi(a)$

22. A regular tetrahedron is full of water and is held with one edge horizontal and the opposite edge inclined at an angle α to the horizon Show that the pressures on the two faces through the horizontal edge are

$$\tfrac{1}{12}\sqrt{3}wa^3(\sin\alpha \pm \sqrt{2}\cos\alpha) \text{ and } \tfrac{1}{12}\sqrt{6}wa^3(\sqrt{2}\sin\alpha \pm \cos\alpha),$$

where a is the length of an edge of the tetrahedron and w is the weight of unit volume of water.

23. A solid right prism whose base is the triangle ABC is kept immersed in a homogeneous liquid with the two bases vertical. If the prism be rotated about a horizontal axis through its c.g , show that

$$P \operatorname{cosec} A + Q \operatorname{cosec} B + R \operatorname{cosec} C$$

remains unaltered, where P, Q, R denote the fluid thrusts on the lateral faces through BC, CA, AB respectively.

22. Determination of the Centre of Pressure of a Plane Area. The term centre of pressure, which is generally written as c p , is defined in Art 17 We shall, at present, discuss the geometrical methods of its determination For that purpose the following theorem would be found useful :

The centre of pressure of a plane area lies vertically beneath the centre of gravity of the superincumbent fluid

If from every point of the perimeter of the area verticals are drawn to meet the free surface (where the pressure is zero), the volume of the fluid enclosed by these lines and the plane area is called the *superincumbent fluid*

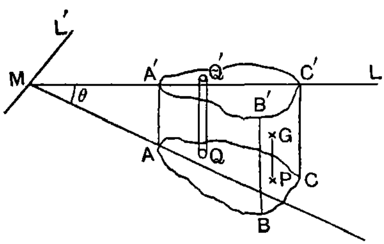

FIG 17

Let ABC be the area and $A'B'C'$ be its projection on the free surface Consider the equilibrium of the superincumbent fluid. The forces acting on it are (1) its weight acting vertically downwards through its c g G ; (2) normal pressure on base ABC which is equal and opposite to the fluid thrust on the area ABC and therefore acting at its c.p., P ; (3) normal pressure on face $A'B'C'$, which is zero by assumption , and (4) pressure on the curved surface, which is horizontal at every point The vertical components of these forces must themselves form a system in equilibrium. They are : the weight of the volume at G and the

vertical component of the fluid thrust on the area acting at P Hence they must act along the same line, so that GP is vertical, which proves the theorem.

23. Let ML' be the intersection (Fig 17) of the plane of the area with the free surface Divide the area ABC into small elements such as α at Q, and draw the cylinder QQ' The pressure on this element $= g\varrho\alpha \; QQ' = g\varrho\alpha \, . \, MQ \sin\theta$ where θ is the angle between the plane ABC and the free surface Therefore the fluid pressure on the area is equal to the system of pressures like $g\varrho\alpha \, . \, MQ \sin\theta$ at Q.

Now, if the area ABC be turned round ML', the angle θ will change to some other value θ' (say), whilst MQ remains unaltered in length for every element Therefore the pressure on every element will be changed in the ratio $\frac{\sin\theta'}{\sin\theta}$, since the pressure on the element at Q in its new position is obviously $g\varrho\alpha \; MQ \sin\theta'$ From statics we know that if the magnitude of every force of a system of parallel forces acting at given points be changed in a constant ratio, the position of the centre of the parallel forces is not affected In this case the original system of pressures has been changed in the ratio $\frac{\sin\theta'}{\sin\theta}$, therefore *the position of the c p of the area does not change relative to the area by the rotation of its plane about ML'.*

The above result enables us to determine the c p. of a vertical area, although there would be no superincumbent liquid in this case Suppose the area to be moved round its intersection with the free surface, and the c.p. of the area can be determined in the inclined position The same point would give the position of the c.p in the vertical position also

Let the area (Fig 17) be turned about ML' till it lies on the free surface, the same point P will denote the c p. of

the area in any position, except the final one So long as the plane of the area is inclined to the horizontal, however small the inclination may be, the above theorem holds. But when the plane lies on the free surface (where the pressure is zero) the pressure vanishes, and so the c p has no meaning for this position

24. We can use the method of Art 22 with advantage in the following cases, the results of which are important and should be remembered

(i) A rectangle $ABCD$, with one side AB in the surface of a homogeneous fluid

The form of the superincumbent fluid is a triangular prism. Therefore its c g. is the centroid G of the middle section HLM Draw GP vertical (i e parallel to ML) to meet HL in P Then P is the c p required

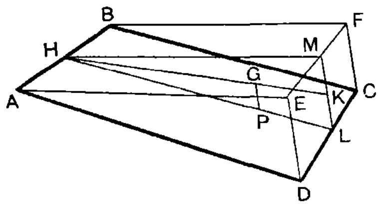

FIG 18

From the figures,

$$HP : PL = HG : GK$$
$$= 2 : 1.$$

Hence the c p divides the line joining the mid-points of horizontal sides in the ratio 2 : 1 Its distance from the surface-line AB is $\frac{2}{3}HL$ or $\frac{2}{3}AD$ . .. . (9)

(ii) A triangle ABC with one side AB in the surface of the liquid

The superincumbent liquid in this case is a tetrahedron Therefore its c g G lies in the plane CDE and divides CF in the ratio 3 : 1, where F is the centroid of the triangle ABD Draw GP vertical to meet CE in P, then P is the c p required Draw FH parallel to GP Since F divides ED in the ratio 1 : 2 and FH is parallel to CD,

$$\therefore\ EH = \tfrac{1}{3}EC, \text{ or } HC = \tfrac{2}{3}EC$$

Again, $HP : PC = FG : GC = 1 : 3$. Therefore

$$CP = \tfrac{3}{4}HC = \tfrac{1}{2}EC.$$

Hence P bisects the median EC of the triangle ABC ... (10)

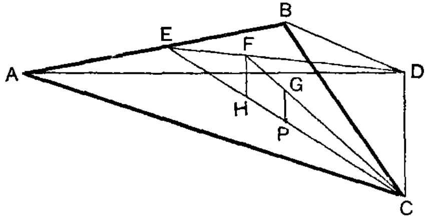

FIG 19

(iii) A triangle ABC with vertex A in the surface of the liquid and the base BC horizontal.

The superincumbent liquid is in the form of a pyramid with A as vertex and the rectangle $BCED$ as the base. Therefore its c g G lies on AH, H being the middle point of KF, the line joining the mid-points of DE and BC. Draw GP vertical meeting AF in P, then P is the c p required. Now,

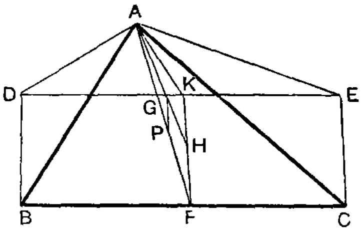

FIG 20

$$AP : PF = AG : GH$$
$$= 3 : 1,$$

$\therefore$ $AP = \tfrac{3}{4}AF$ (the median of the triangle ABC) .. (11)

25. As the c p is the centre of a system of parallel forces (pressures on elements of area), we can utilise methods analogous to the determination of the c g (given in Statics) of a body. For example.

Given the position of the c p and the pressure on each of the two portions into which an area is divided, to find the c p of the whole area

Let P_1, P_2 be the c p. and p_1, p_2 be the pressures on the two portions A and B respectively That is, we are given that the pressure p_1 on A acts at P_1 and the pressure p_2 on B acts at P_2, and we are required to find where the total pressure, viz (p_1+p_2), on the whole area will act This total pressure is clearly the resultant of p_1 and p_2, and therefore acts at P (on the straight line joining P_1, P_2) such that $PP_1 : PP_2 = p_2 \quad p_1$. This gives the position of P

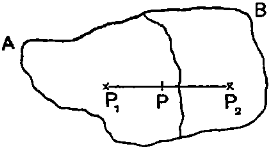

FIG 21

Again, *if the c p and the pressures on the whole area and on one of the parts be given, to find the c p of the other part*

Let P, P_1 be the c p , and p, p_1 be the pressures on the whole area and the part A respectively Then the pressure p (on the whole area) will be the resultant of the pressure p_1 (on A) and the pressure $p-p_1$ (on the part B) , let the last act at P_2 Then P_1PP_2 is a straight line and

$$PP_2 \,.\, PP_1 = p_1 \quad p-p_1,$$

thus giving the c p required

26. *Given the positions of the c p and the c g of an area in a given position, to find the c p of the area when it is shifted parallel to itself to a greater depth*

Let G be the c g and P the c p of the area in the first position and let z and ζ denote their depths below the free surface of the liquid , we shall assume that there is no pressure at the free surface

Divide the area into elements, a_1, a_2, ... a_n ... and let h_n denote the depth of the element a_n Then the pressure on this element is $g\rho a_n h_n$ and the resultant of all such pressures is the pressure $g\rho Az$ acting at P (Art 18).

Next, take the area to a further depth d, the depth of every element being increased by d , so that the pressure on the element a_n is now $g\rho a_n(h_n+d)$. This pressure can

be divided into two parts, $g\varrho a_n h_n$ and $g\varrho d a_n$. Similarly for every other element. Thus the system of pressures is divided into two other systems:

(i) pressures $g\varrho a_n h_n$ acting on a_n, and
(ii) pressures $g\varrho d a_n$ acting on a_n.

The resultant of the first is $g\varrho Az$ acting at P (by hypothesis), whilst that of the second is $g\varrho dA$ acting at G (since $g\varrho d a_n$ would denote the weight of the element a_n if $g\varrho d$ were the weight per unit area and the total weight of the elements would then act at the c g of the area). The resultant of these two resultants will be the total pressure on the area at the new depth. Hence the rule of compounding like parallel forces gives that the new position, P', of the c p. is on GP and is such that

$$GP' : P'P = g\varrho Az : g\varrho dA = z : d \quad \ldots\ldots\ldots \quad (12)$$

$\therefore$ depth of P' (in the new position of the area)

$$= \frac{z(h+d) + d(z+d)}{z+d}, \quad \ldots\ldots\ldots \quad (13)$$

since the depths of G and P are now $z+d$ and $h+d$ respectively.

From (12), we get by componendo, $\dfrac{GP'}{GP} = \dfrac{z}{z+d}$.

This shows that as d increases, GP' diminishes, but it is never zero, for its vanishing would require d to be infinite. Thus the c p. of an area is always below its c g and their distance diminishes as the depth of the c g increases.

If the area is lowered so that the c g describes a vertical line it is easy to see that the c p approaches this line as the depth is increased, that is, the vertical line is an asymptote to the locus of the c p in space.

27. In the following worked out examples, the plane of the area is taken to be vertical for the sake of simplicity.

The same formulae will give the c p of the areas in inclined positions if the vertical depths are replaced by slant depths measured down the plane

Ex 1 Find the c p of a triangle at any depth, one side being horizontal

There may be two cases according as the horizontal side is uppermost or lowermost We shall consider the second case

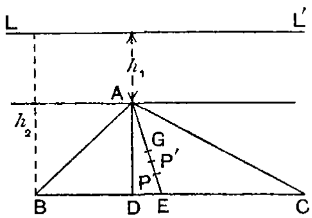

FIG 22

Let the depth of A below the surface be h_1 and that of B or C be h_2

If A were on the surface, the c p of the triangle would have been on the median AE so that $AP=\frac{3}{4}AE$ In this position $z=$ depth of the c g , $G=\frac{2}{3}AD=\frac{2}{3}(h_2-h_1)$ The triangle requires to be lowered through a depth h_1 in order to bring it to the given position Therefore the c p , P', lies on GP (Art 26), so that

$$GP' : GP=\tfrac{2}{3}(h_2-h_1) : h_1+\tfrac{2}{3}(h_2-h_1)$$

$$=2(h_2-h_1) : 2h_2+h_1$$

$$\therefore\ GP'=\frac{h_2-h_1}{2h_2+h_1}\cdot\frac{1}{6}AE.$$

$\therefore$ depth of $P'=$ depth of $G+$ projection of GP' on AD

$$=\frac{2h_2+h_1}{3}+\frac{1}{6}\frac{(h_2-h_1)^2}{2h_2+h_1}$$

$$=\frac{1}{2}\frac{3h_2^2+h_1^2+2h_2h_1}{2h_2+h_1}$$

This result could have been directly obtained from the formula (13) When the side BC is uppermost we can arrive at the very formula by a similar process

Ex. 2 Find the c.p. of a trapezium, one of its parallel sides being on the surface.

Divide the area into two triangles ABC and ADC. Let $AB=a$, $CD=b$, altitude of the trapezium $=h$.

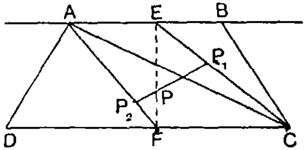

FIG. 23

Pressure on $\triangle ABC$ is $\frac{1}{6}g\varrho ah^2$, acting at P_1, the mid-point of CE (Arts. 18 and 24).

Pressure on $\triangle ACD$ is $\frac{1}{3}g\varrho bh^2$, acting at P_2 which divides AF in the ratio 3 : 1.

$\therefore$ the c.p., P, of the trapezium is on P_1P_2 (Art. 25), such that

$$P_1P : PP_2 = \tfrac{1}{3}g\varrho bh^2 : \tfrac{1}{6}g\varrho ah^2 = 2b : a$$

The depths of P_1 and P_2 are $\frac{1}{2}h$ and $\frac{3}{4}h$ respectively. Therefore the depth of P is given by

$$\frac{2b\cdot\frac{3}{4}h + a\cdot\frac{1}{2}h}{2b+a}, \text{ or } \frac{h}{2}\cdot\frac{a+3b}{a+2b}$$

Note. If we divide the area into thin horizontal strips, the pressures on which act at their middle points (since the pressure is uniform over each strip), we get a system of pressures acting at the middle points of the strips, *i.e.* on points on the line EF. Therefore their resultant must act at a point on EF, that is to say, the c.p. of the area must be on EF. So EF and P_1P_2 necessarily intersect at P.

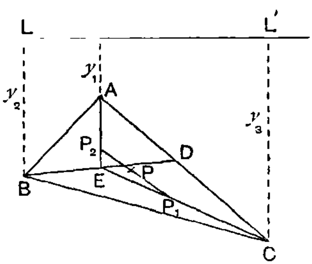

FIG. 24

Ex. 3 If the depths of the vertices A, B, C of a triangle be y_1, y_2, y_3, find the depth of the c.p. of the triangle.

First method. Divide the triangle into two others by the horizontal line BD.

Pressure on $\triangle ABD = g\varrho \cdot \frac{1}{2}BD(y_2 - y_1) \cdot \frac{1}{3}(y_1 + 2y_2)$

$$= \tfrac{1}{6}g\varrho BD \cdot (y_2 - y_1)(2y_2 + y_1),$$

and the depth of its c p , P_2,

$$= \frac{1}{2} \; \frac{3y_2^2 + y_1^2 + 2y_1y_2}{2y_2 + y_1},$$

from Ex. 1.

Similarly, the pressure on $\triangle BDC$

$$= \tfrac{1}{6}g\varrho BD \cdot (y_3 - y_2)(y_3 + 2y_2);$$

and the depth of its c p , P_1,

$$= \frac{1}{2} \cdot \frac{3y_2^2 + y_3^2 + 2y_3y_2}{2y_2 + y_3}.$$

$\therefore$ by Art 25, the c p , P, of the whole $\triangle ABC$ divides P_1P_2 in the ratio

$$(y_2 - y_1)(2y_2 + y_1) : (y_3 - y_2)(y_3 + 2y_2).$$

Hence the depth of P

$$= [\tfrac{1}{2}(y_2 - y_1)(3y_2^2 + y_1^2 + 2y_1y_2) + \tfrac{1}{2}(y_3 - y_2)(3y_2^2 + y_3^2 + 2y_3y_2)] \div [(y_2 - y_1)(2y_2 + y_1) + (y_3 - y_2)(y_3 + 2y_2)]$$

$$= \frac{1}{2} \; \frac{y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_2y_3 + y_3y_1}{y_1 + y_2 + y_3}$$

after removing the common factor $(y_3 - y_1)$.

Second method Let A be the highest vertex Take AB, AC as axes of x and y. Consider an element, α, of the area at Q whose coordinates are (x, y). Draw QK parallel to BA , then $QK = x$, $AK = y$. Let the inclinations of the axes of x and y to the vertical be θ and β respectively. Draw the medians AD, BE and CF Also draw KH, AM perpendiculars to the vertical QL drawn from Q to meet the surface. Let A denote the area of the $\triangle ABC$.

Pressure on element at Q

$$= g\varrho a \, . \, QL = g\varrho a \, (QH + HM + ML)$$

$$= g\varrho a \, . \, (x \cos \theta + y \cos \beta + y_1)$$

Similarly for pressures on other elements, we thus get a system of pressures or parallel forces We can divide this system into the three following systems (of parallel forces) :

(1) System of forces $g\varrho ax \cos \theta$ acting at every element a at (x, y) To find the resultant of this system let us assume

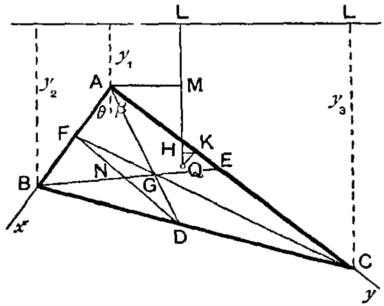

FIG 25

the triangle ABC immersed in a liquid with AC in the surface Then the pressure on the element a would have been $g\varrho' xa \sin BAC$, since the distance of Q from AC is $x \sin BAC$ Let the density ϱ' of the liquid be

$$= \varrho \cos \theta / \sin BAC \;\; (= \text{a constant}).$$

Thus we have the pressure on the element $= g\varrho ax \cos \theta$ The resultant liquid pressure would therefore be

$$g\varrho' \;\; A \, . \, \frac{c}{3} \sin BAC \;\text{ or }\; g\varrho A \, \frac{c}{3} \cos \theta$$

at the middle point N of the median BE (Arts 18 and 24)

This resultant force at N can be replaced by two equal forces, each $= \frac{1}{6} g\varrho Ac \cos \theta$, at D and at F.

(2) System of forces $g\varrho a \cos\beta \; y$ at every element, which can be reduced to two equal forces, each

$$=\tfrac{1}{6}g\varrho Ab\cos\beta,$$

at D and at E, as in the case of the first system

(3) System of forces $g\varrho a y_1$ at every element, which is equivalent to $g\varrho y_1 A$ acting at the c g of the $\triangle ABC$ (cf. Art 26), which again can be replaced by three equal forces, each $=\tfrac{1}{3}g\varrho y_1 A$, at D, at E and at F

Thus the whole system reduces to three forces, viz $\tfrac{1}{3}g\varrho A\,(y_1+\tfrac{1}{2}c\cos\theta+\tfrac{1}{2}b\cos\beta)$ at D, $\tfrac{1}{3}g\varrho A\,(y_1+\tfrac{1}{2}b\cos\beta)$ at E and $\tfrac{1}{3}g\varrho A\,(y_1+\tfrac{1}{2}c\cos\theta)$ at F

But $c\cos\theta=y_2-y_1$ and $b\cos\beta=y_3-y_1$

$\therefore$ forces at D, E and F are

$$\frac{1}{3}g\varrho A\left(\frac{y_2+y_3}{2}\right),\ \frac{1}{3}g\varrho A\left(\frac{y_3+y_1}{2}\right),\ \frac{1}{3}g\varrho A\ \left(\frac{y_1+y_2}{2}\right)$$

respectively, *i e* they are $\tfrac{1}{3}g\varrho A$ times their respective depths Hence the following important result.

The whole system of fluid pressures on the elements of a triangle ABC (and therefore the total fluid pressure on the triangle) is equivalent to three parallel forces at D, E, F, equal to $\tfrac{1}{3}g\varrho A$ times the depths of the respective points of action Therefore the c p of the triangle must coincide with the centre of the three forces at D E, F

$\therefore$ the depth of the c p

$$=\frac{\left(\frac{y_2+y_3}{2}\right)^2+\left(\frac{y_3+y_1}{2}\right)^2+\left(\frac{y_1+y_2}{2}\right)^2}{\frac{y_2+y_3}{2}+\frac{y_3+y_1}{2}+\frac{y_1+y_2}{2}},$$

removing the common factor $\tfrac{1}{3}g\varrho A$ from above and below,

or $$=\frac{1}{2}\,\frac{y_1^2+y_2^2+y_3^2+y_1y_2+y_2y_3+y_3y_1}{y_1+y_2+y_3}.$$

Note. If x_1, x_2, x_3 be the abscissae of A, B, C referred to some suitable line in the plane of the triangle perpendicular to LL', the above theorem gives the abscissa of the c p as

$$\left[\left(\frac{y_1+y_2}{2}\right)\left(\frac{x_1+x_2}{2}\right)+\left(\frac{y_3+y_1}{2}\right)\left(\frac{x_3+x_1}{2}\right)+\left(\frac{y_2+y_3}{2}\right)\left(\frac{x_3+x_2}{2}\right)\right]$$
$$\div\left[\left(\frac{y_1+y_2}{2}\right)+\left(\frac{y_3+y_1}{2}\right)+\left(\frac{y_2+y_3}{2}\right)\right],$$

or
$$\frac{1}{4}\left[x_1+x_2+x_3+\frac{x_1y_1+x_2y_2+x_3y_3}{y_1+y_2+y_3}\right]$$

The ordinate (or depth) of the c p can be put in a similar form :

$$\frac{1}{4}\left[y_1+y_2+y_3+\frac{y_1^2+y_2^2+y_3^2}{y_1+y_2+y_3}\right]$$

Another and shorter method of deducing these results will be given in the next chapter

28. In the above investigations the effect of the atmosphere is neglected If this is to be taken into consideration we first determine the effective surface of the liquid (Art 11) and determine the c p of the area by taking this effective surface as the free surface, as in Art 26

EXAMPLES. 3.

Find the positions of the c p of the following areas

1 A parallelogram with one side in the surface and show that it is given by the same construction as that of a rectangle

2. A rectangle at any depth, sides a, b, one pair is horizontal

3. A parallelogram with one pair of sides horizontal and at depths h_1, h_2 below the surface of the liquid and its plane vertical.

• 4. A trapezium with parallel sides of lengths a, b, at depths h_1, h_2 below the surface, its plane being vertical.

5. A trapezium is placed with one of the oblique sides in the surface when the two parallel sides (of lengths b and d) are vertical Find the depth of the c.p. of the area

6. Show that the horizontal line drawn through the c.p. of a rectangle, one of whose sides is in the surface, divides the rectangle into two parts the pressures on which are as 4 : 5.

In the case of a triangle with a side in the surface, show that a similar line divides the area into two portions the pressures on which are equal.

7. P is the c.p. of the rectangle $ABCD$, the side AB being in the surface. Prove that the line through A and P divides the area into two portions the pressures on which are equal.

8. A rhombus $ABCD$ is completely immersed with the diagonal AC vertical and A on the surface. Prove that its c.p. divides AC in the ratio 7 : 5.

9. A rhombus is totally immersed with one diagonal vertical and its centre at depth h. Prove that the c.p. is $\frac{a^2}{24h}$ below the centre where a is the length of the vertical diagonal.

10. A parallelogram $ABCD$ is immersed in a liquid with A in the surface and BD horizontal; prove that its c.p. (P) lies on AC and that $AP : AC = 7 : 12$.

11. A vertical sluice gate, which is a square of side a, has its upper edge hinged at the surface of water. Neglecting the effect of the atmosphere, find the force which must be applied at the middle point of the base to keep the gate in place.

12. An isosceles triangular lamina of height b is immersed vertically in water with the vertex in the surface and base horizontal. If h be the height of the water barometer, prove that the depth of the c.p. below the surface of water is

$$\frac{b}{2} \cdot \frac{4h+3b}{3h+2b}.$$

13. A rectangle is immersed with one side in the surface. Show how to divide the area by a horizontal line so that the c.p. of each part will be at the same distance from this horizontal line.

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14. A cubical box, standing on a horizontal plane, has one of its vertical sides capable of revolving about a hinge at the bottom. If a portion of a liquid equal in volume to one-fourth the capacity of the cube be put in the vessel, the loose side rests at an inclination (inwards) of 45°. If no liquid escapes, compare the weight of the loose side with the weight of the liquid in the vessel.

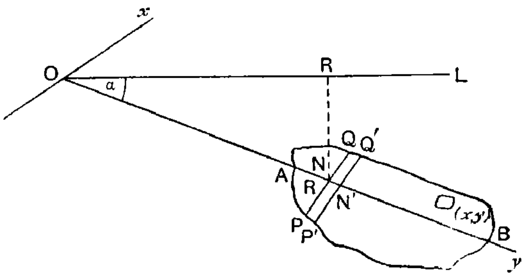
15. $ABCD$ is a trapezium whose parallel sides are AD and BC. It is immersed in a liquid with its plane vertical so that A, B, C, D are at depths α, β, γ, δ respectively. Find the depth of its c p.

16. A quadrilateral $ABCD$ is immersed with the side AB in the surface and its plane vertical If a, b denote the depths of C, D and x, y denote the depths of its c.g and c.p. respectively, prove that $6xy + ab = 3x(a + b)$.

CHAPTER IV

CENTRE OF PRESSURE (*Continued*)

29. Determination of the c.p. of a plane area by the methods of calculus. We shall first consider the area immersed in homogeneous liquid We take the intersection of the plane of the area with the free surface of the liquid



as the x-axis and any suitable line in the plane as the y-axis [cf. Art 20] Denote the inclination of the plane to the horizontal by α Divide the area into thin horizontal strips, such as $PQQ'P'$, let $ON=y$, $NN'=dy$ Then the area, dS, of the element $=PQdy=(x_Q-x_P)dy$, where x_Q and x_P can be obtained from the knowledge of the equation of the perimeter of the area, it is to be noted that x_P is negative (here)

The pressure on the element $=g\rho y \sin\alpha \,.\, PQdy$

$$\therefore \text{ the total pressure} = g\rho \sin\alpha \int_A^B y \quad PQdy$$

$$= g\rho \sin\alpha \quad \bar{y}S,$$

as in Art. 20, where $\bar{y}$ is the distance of the c g of the area from Ox, and S the measure of the area.

Let (X, Y) be the coordinates of the c p of the area. The resultant pressure whose value is just determined, acting at (X, Y), is the resultant of the system of pressures like $g\varrho \sin\alpha \cdot yPQdy$ or $g\varrho \sin\alpha \cdot ydS$, acting at R the middle point of PQ (since the pressure is uniform over this element) The coordinates of R are $[\frac{1}{2}(x_P + x_Q), y]$ Hence equating the moments about Ox, we get

$$Y \cdot y . Sg\varrho \sin\alpha = \Sigma g\varrho \sin\alpha . y^2PQdy$$

$$= \int_A^B g\varrho \sin\alpha \cdot y^2PQdy,$$

whence, removing the common factor $g\varrho \sin\alpha$,

$$Y = \frac{\int_A^B y^2PQdy}{yS} \quad \ldots\ldots\ldots (1)$$

Similarly, by equating the moments about Oy, we arrive at

$$X = \frac{\int_A^B \frac{1}{2}(x_P + x_Q) \cdot yPQdy}{yS} \quad \ldots\ldots\ldots (2)$$

The formulae (1) and (2) give the position of the c p relative to the area They do not involve α, depending only on the distances from Ox and Oy Therefore, if α be changed whilst the latter remain unaltered, as would be the case when the area is turned round Ox, the position of the c p relative to the area remains the same. This result was arrived at in Art 23 in a different manner.

30. Next, let us take the atmospheric pressure into consideration The pressure on the strip PQ' will now be equal to $(\Pi + g\varrho y \sin\alpha)PQdy$, if Π be the pressure due to atmosphere

$$\therefore \text{ total pressure} = \int_A^B (\Pi + g\varrho y \sin\alpha)PQdy$$

$$= (\Pi + g\varrho y \sin\alpha)S.$$

Taking moments about Oy and Ox, as before, we get

$$X = \frac{\int_A^B \frac{1}{2}(x_P + x_Q)(\Pi + g\varrho y \sin \alpha) PQ dy}{(\Pi + g\varrho \bar{y} \sin \alpha) S} \quad \text{. (3)}$$

$$Y = \frac{\int_A^B y(\Pi + g\varrho y \sin \alpha) PQ dy}{(\Pi + g\varrho \bar{y} \sin \alpha) S} \quad \text{.................. .(4)}$$

If, however, there be strata of n different liquids of densities $\varrho_1, \varrho_2, \dots \varrho_n$, and of slant depths $h_1, h_2, \dots h_n$ from the top, and the area be wholly in contact with the nth liquid, the coordinates of the c p will be given by

$$Y \int_A^B (\Pi + g\varrho_1 h_1 \sin \alpha + g\varrho_2 h_2 \sin \alpha + \dots + g\varrho_n y \sin \alpha) PQ dy$$

$$= \int_A^B y(\Pi + g\varrho_1 h_1 \sin \alpha + \dots + g\varrho_n y \sin \alpha) PQ dy, \quad \text{.........(5)}$$

and a similar equation for X, the axis of x being taken on the upper surface of the nth liquid These can be deduced from (3) and (4) by replacing (therein) Π by

$$\Pi + g\varrho_1 h_1 \sin \alpha + g\varrho_2 h_2 \sin \alpha + \dots + g\varrho_{n-1} h_{n-1} \sin \alpha,$$

which is now the pressure at the surface xOL

Lastly, if the area be in contact with more than one liquid, each integral in (5) is to be replaced by the sum of two or more definite integrals, each of the latter involving each portion of the area in contact with one liquid only.

It also becomes necessary at times to subdivide the interval (from A to B), *i e.* to evaluate the definite integrals over each sub-interval separately and then to take their sum. A little common sense as well as a little practice will be a good guide to find out whether such a method of evaluation is to be followed or not

31. In a number of examples we are able to deduce, from geometrical considerations or otherwise, that the c.p.

lies on a definite line. It is then sufficient to determine only one coordinate by means of the formulae given above, the other being then readily deduced (see the examples below)

A few simple cases are given below as illustrations The beginner is advised not to use the above formulae, substituting therein the values of the particular cases for evaluation, they should instead proceed as has been done in the previous articles before arriving at these formulae. Atmospheric pressure is generally not taken into account unless it is specially mentioned to do so.

Ex. 1 A rectangle immersed in a homogeneous liquid with two sides horizontal and at depths h_1 and h_2 ($> h_1$), its plane being vertical.

Here we take Oy through the middle points E and F of AB and CD respectively, since it is an axis of symmetry it is clear that the c p will lie on it Therefore $X = 0$ (as can actually be shown), and we have to evaluate Y only

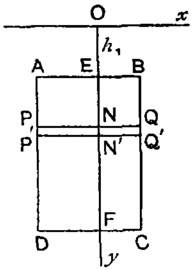


FIG 26

Let $AB = a$, divide the area into strips like $PQQ'P'$ Let $ON = y$, $NN' = dy$, $OE = h_1$, $OF = h_2$.

The pressure on the strip $= g\varrho yady$, acting at N,

$\therefore$ moment of this pressure about $Ox = g\varrho y^2ady$;

$$\therefore \quad Y\int_{h_1}^{h_2} g\varrho yady = \int_{h_1}^{h_2} g\varrho y^2ady,$$

whence $$Y = \frac{2}{3}\frac{h_1^2 + h_1h_2 + h_2^2}{h_1 + h_2}.$$

If AB be in the surface, $h_1 = 0$, in which case $Y = \frac{2}{3}h_2$, as in Art 24 (9)

Ex 2. A triangle with one side horizontal at any depth.

Since the inclination of the plane of the triangle to the free surface is immaterial, we might take the area to be vertical. Let $BC=a$ and the depths of A and B be h_1, h_2 respectively

Let the triangular area be divided into horizontal strips, like $PQQ'P'$, the pressure on which acts obviously at the mid-point R Thus we get a system of pressures acting at

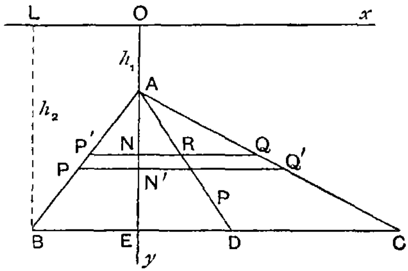


FIG 27

points which lie on the median AD Hence their centre or their c.p lies on AD itself We need, therefore, only determine one coordinate, viz Y of the c.p

Let $ON=y$ $NN'=dy$ Then $\dfrac{PQ}{BC}=\dfrac{AN}{AE}=\dfrac{y-h_1}{h_2-h_1}$.

$$\therefore\ PQ=\frac{a}{h_2-h_1}(y-h_1)=\lambda(y-h_1),\ \text{say.}$$

Now, pressure on the strip $=g\varrho y\ \ \lambda(y-h_1)dy$, and its moment about $Ox=g\varrho y^2\ \ \lambda(y-h_1)dy$

$$\therefore\ Y\int_{h_1}^{h_2} g\varrho\lambda y(y-h_1)dy=\int_{h_1}^{h_2} g\varrho\lambda y^2(y-h_1)dy,$$

whence $$Y=\frac{1}{2}\ \frac{3h_2^2+2h_2h_1+h_1^2}{2h_2+h_1}.$$

Let the c p, P, divide AD in the ratio $m . n$. Then, since the ordinates of A and D are h_1 and h_2, the ordinate

of P is $\frac{nh_1 + mh_2}{m+n}$. Comparing this with the above result we get

$$nh_1 + mh_2 = 3h_2^2 + 2h_2h_1 + h_1^2,$$
$$m + n = 4h_2 + 2h_1,$$

whence $m = 3h_2 + h_1$, $n = h_2 + h_1$ Thus the c p divides the median in the ratio $3h_2 + h_1 : h_2 + h_1$. The abscissa of P can now be easily calculated in terms of those of A, B and C

✓*Ex* 3 A vertical circle of radius a, totally immersed with its centre at a depth h below the surface

Take the vertical line through the centre as axis of y Then since this line bisects all horizontal chords, the c p must lie on this axis $\therefore X = 0$, and we have to determine Y only.

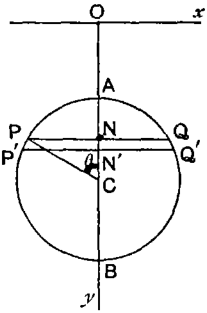


FIG 28

Let $PQQ'P'$ be the elementary strip, $ON = y$ and $NN' = dy$ Then

$$PQ = 2\sqrt{a^2 - (h-y)^2}$$

Pressure on the strip

$$= g\rho y \cdot 2\sqrt{a^2 - (h-y)^2}\, dy,$$

and its moment about

$$Ox = g\rho y^2 \cdot 2\sqrt{a^2 - (h-y)^2}\, dy$$

$$\therefore\ Y \int_{h-a}^{h+a} 2g\rho y \sqrt{a^2 - (h-y)^2}\, dy$$
$$= \int_{h-a}^{h+a} 2g\rho y^2 \sqrt{a^2 - (h-y)^2}\, dy,$$

or $\quad Y \displaystyle\int_0^{\pi} 2g\rho a^2 \sin^2\theta\,(h - a\cos\theta)\, d\theta$

$$= \int_0^{\pi} 2g\rho a^2 \sin^2\theta\,(h - a\cos\theta)^2 d\theta,$$

where $\angle ACP = \theta$ and $\angle PCP' = d\theta$, whence

$$Y = h + \frac{a^2}{4h}$$

Thus the c.p. of a circle is at a distance $\frac{a^2}{4h}$ below the centre.

Ex. 4. A circle whose radius is a and plane vertical is immersed in water with one point A of its circumference at a depth $b\,(>2a)$. If the circle be turned in its own plane about the point A, find the locus of the c.p (i) relative to the area, (ii) in space

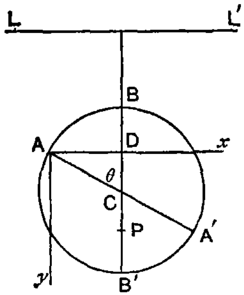


FIG 29

The depth $b>2a$ ensures that the circle will be completely immersed in all positions Suppose that, in any position, the vertical diameter is inclined at an angle θ with CA Then the depth of C (in this position) = depth of $A+CD$

$$=b+a\cos\theta.$$

$\therefore$ the c p , P, lies on CB' such that $CP=\dfrac{a^2}{4(b+a\cos\theta)}$, by Ex 3 Hence, if we take C as pole, CA' as the initial line and (r, θ) as the polar coordinates of P, we shall have

$$CP=r=\frac{a^2}{4(b+a\cos\theta)}, \text{ or } \frac{a^2}{4br}=1+\frac{a}{b}\cos\theta,$$

which shows that the locus of P relative to the area is an ellipse with C as focus, the major axis along ACA', eccentricity $\frac{a}{b}$ and latus rectum $\frac{a^2}{2b}$

To determine the form of the locus in space, take A as origin and the horizontal and the vertical lines as coordinate axes If P be (x, y), we have

$$\left.\begin{aligned} x&=a\sin\theta,\\ y&=a\cos\theta+\frac{a^2}{4(b+a\cos\theta)}.\end{aligned}\right\}$$

These constitute the equations of the locus

32. Those who are familiar with *moments* and *products of inertia* will find the following formulae for the c p most convenient

Divide the area (see fig 13, Art 20) into small elements, one of which is of area σ surrounding a point whose coordinates are (x, y) Then the pressure on this element is $g\varrho y \sin\alpha \;\sigma$, and moments of this pressure about Ox and Oy are $g\varrho y^2 \sin\alpha \,.\, \sigma$ and $g\varrho xy \sin\alpha \,.\, \sigma$ respectively. Therefore

$$X = \frac{\Sigma g\varrho xy \sin\alpha \quad \sigma}{\Sigma g\varrho y \sin\alpha \quad \sigma} = \frac{\Sigma xy\sigma}{\Sigma y\sigma}$$

$$= \frac{\text{product of inertia of the area about } Ox, Oy}{\text{moment of the area about } Ox} \quad \ldots. \text{(6)}$$

$$Y = \frac{\Sigma g\varrho y^2 \sin\alpha \quad \sigma}{\Sigma g\varrho y \sin\alpha \quad \sigma} = \frac{\Sigma y^2\sigma}{\Sigma y\sigma}$$

$$= \frac{\text{moment of inertia of the area about } Ox}{\text{moment of the area about } Ox} \quad \ldots\ldots\ldots\text{(7)}$$

The denominator in these formulae is clearly $\bar{y}S$, and the numerators can be calculated with the help of the theorems of inertia.

33. For the sake of ready reference some rules about the moments and products of inertia are given below :

(i) If one of the axes, Ox or Oy, be an axis of symmetry, the product of inertia is zero.

(ii) Moment of inertia of a rectangle, sides a and b, about a line bisecting the sides b at right angles $= \frac{b^2}{12} \,.\, A$, A denoting the measure of the area.

(iii) Moment of inertia of an ellipse, semi-axes a and b, about the axis $a = \frac{b^2}{4} \,.\, A$.

(iv) Moment of inertia of a circle about any diameter $= \frac{a^2}{4} \,.\, A$

(v) Moment of inertia of a compound area = sum of the moments of inertia of its various parts about the same line.

(vi) Moment of inertia about an axis = moment of inertia about a parallel axis through the c g of the mass + moment of inertia of the whole mass collected at the c g about the original axis

Similar theorem for the product of inertia This is known as *the theorem of parallel axes*

(vii) Let I_1, I_2 and P denote the moments of inertia about Ox, Oy and the product of inertia about the axes, θ be the angle which a line OQ makes with Ox Then moment of inertia about OQ

$$= I_1 \cos^2\theta + I_2 \sin^2\theta - 2P \cos\theta \sin\theta$$

(viii) The moments and products of inertia of a triangle about any axes are the same as those of three equal particles (each of one-third the mass of the whole triangle) placed at the middle points of the sides This system of particles and the triangle have obviously the same mass and the same centre of gravity, and therefore the same moment of masses about any axis Such systems are called *equimomental systems* This result is very useful

34. Applications.

(1) *Rectangle* with two sides horizontal and at depths h_1 and h_2 below the surface Depth of its c g $= \frac{1}{2}(h_1 + h_2)$, moment of inertia about Ox (see fig 26)

$$= A\left[\frac{(h_2 - h_1)^2}{12} + \frac{(h_2 + h_1)^2}{4}\right] \text{ by rules (ii) and (vi)}$$

$$= \frac{A}{3}(h_1^2 + h_2^2 + h_1 h_2) \quad \text{Therefore from (7),}$$

$$Y = \frac{\frac{1}{3}A(h_1^2 + h_2^2 + h_1 h_2)}{\frac{1}{2}(h_1 + h_2)A} = \frac{2}{3}\,\frac{h_1^2 + h_2^2 + h_1 h_2}{h_1 + h_2}, \text{ as before.}$$

(2) *Circle* of radius a and its centre at depth h

Moment of inertia of the circle about Ox (see fig. 28)

$=A\left(\frac{a^2}{4}+h^2\right)$, by rules (iv) and (vi)

$$\text{from (7),} \quad Y=\frac{A\left(h^2+\frac{a^2}{4}\right)}{Ah}=h+\frac{a^2}{4h}$$

Since, by rule (i) the product of inertia is zero, we have $X=0$

(3) *Triangle* whose vertices A, B, C are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

Let D, E, F be the middle points of BC CA AB respectively Then the coordinates of D are $\left[\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right]$, of E $\left[\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right]$ and of F $\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right]$

Moment of inertia of the triangle about Ox = moment of inertia of three particles each equal to $\frac{1}{3}A$, at D, E, F, where A denotes the area of the triangle,

$$=\frac{A}{3}\left[\left(\frac{y_2+y_3}{2}\right)^2+\left(\frac{y_3+y_1}{2}\right)^2+\left(\frac{y_1+y_2}{2}\right)^2\right]$$

$$=\frac{A}{6}\left[y_1^2+y_2^2+y_3^2+y_1y_2+y_2y_3+y_3y_1\right].$$

And the moment of the area = the moment of the three particles at D, E, F

$$=\frac{A}{3}\left[\frac{y_2+y_3}{2}+\frac{y_3+y_1}{2}+\frac{y_1+y_2}{2}\right]$$

$$=\frac{A}{3}\left[y_1+y_2+y_3\right].$$

$$\therefore \text{ from (7),} \quad Y=\frac{1}{2}\,\frac{y_1^2+y_2^2+y_3^2+y_1y_2+y_2y_3+y_3y_1}{y_1+y_2+y_3}.$$

Again, the product of inertia of the triangle about Ox, Oy = the product of inertia of the three particles at D, E, F

$$=\frac{A}{3}\left[\left(\frac{y_2+y_3}{2}\right)\left(\frac{x_2+x_3}{2}\right)+\left(\frac{y_3+y_1}{2}\right)\left(\frac{x_3+x_1}{2}\right)\right.$$
$$\left.+\left(\frac{y_1+y_2}{2}\right)\left(\frac{x_1+x_2}{2}\right)\right]$$

$\therefore$ from (6), after a little simplification,

$$X=\frac{1}{4}\left[x_1+x_2+x_3+\frac{x_1y_1+x_2y_2+x_3y_3}{y_1+y_2+y_3}\right].$$

These results were obtained in Art 27.

If parallel forces, proportional to the depths of the respective points, were acting at D, E, F, i e if forces proportional to $\frac{y_2+y_3}{2}$, $\frac{y_3+y_1}{2}$, $\frac{y_1+y_2}{2}$ were acting at D, E, F, their centre would have precisely been given by (X, Y) Hence the result

The centre of pressure of a triangle is the centre of parallel forces acting at the middle points of the sides and proportional to the depths of these points

This was also obtained in Art 27 by a somewhat lengthy process

(4) *Parallelogram* whose vertices A, B, C, D are (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) It is clear that these coordinates are not all independent The relations connecting them are $x_1+x_3=x_2+x_4$ and $y_1+y_3=y_2+y_4$ as the middle points of AC and BD are coincident

Moment of inertia of the parallelogram = sum of moments of inertia of the triangles ABC, ACD

$$=\frac{A}{6}\left[\left(\frac{y_1+y_2}{2}\right)^2+\left(\frac{y_2+y_3}{2}\right)^2+2\left(\frac{y_1+y_3}{2}\right)^2\right.$$
$$\left.+\left(\frac{y_3+y_4}{2}\right)^2+\left(\frac{y_4+y_1}{2}\right)^2\right]$$

or, simplifying with the help of the above relations

$$=\frac{A}{6}[y_1^2+y_2^2+y_3^2+y_4^2+y_1y_3+y_2y_4],$$

where A denotes the area of the parallelogram.

And the moment of the area = sum of moments of the triangles ABC, ACD

$$=\frac{A}{6}\left[\left(\frac{y_1+y_2}{2}\right)+\left(\frac{y_2+y_3}{2}\right)+\dots\right]$$

$$=\frac{A}{2}(y_1+y_3) \text{ or } \frac{A}{2}(y_2+y_4)$$

The last result could have been obtained directly, since the c g of the figure is at the intersection of the diagonals. The value of Y can now be calculated Similar expressions will give X

EXAMPLES. 4.

Find the centres of pressure of

1. A semi-circular area, the diameter being horizontal and at depth b, the radius being a.

2. A quadrant of a circle with one bounding radius horizontal and at depth b, the radius being a

3. An ellipse, completely immersed, with the major axis horizontal and at depth h

4. An ellipse, completely immersed, with the major axis inclined at an angle θ to the horizontal, the centre being at depth h Also show that the c.p lies on the diameter conjugate to the horizontal diameter

5. A semi-ellipse having its bounding diameter (not one of the principal axes) horizontal and at depth h

6. A segment of a parabola bounded by a double ordinate with axis vertical and vertex downwards at a depth h

7 A quadrant of a circle with the centre in the surface and bounding radii equally inclined to the vertical.

8 The area bounded by two concentric semi-circles with their common bounding diameter in the surface.

9. The area included between an hyperbola, one asymptote and two horizontal lines, the other asymptote to the curve lying in the surface

10. An area bounded by the curve $ay^2 = x^3$, an abscissa of length h and the ordinate at its extremity, is placed in water with the ordinate in the surface. Prove that the depth of the c.p. is $\frac{4}{9}h$.

11. A square lamina, of side a, has a portion of it in the form of the inscribed circle removed from it and the remaining figure is immersed vertically in water with one side of the square in the surface Find the depth of the c p.

12. A quadrilateral lamina $ABCD$ whose sides AB, CD are parallel is immersed vertically in a liquid with AB in the surface. Prove that the c p will be at the intersection of the diagonals if $AB = \sqrt{3}\ CD$

13. If the c p. of a triangle ABC, completely immersed, coincide with its circumcentre, show that the depths of the angular points are as

$$1 - 2 \cot B \cot C \quad 1 - 2 \cot C \cot A \quad 1 - 2 \cot A \cot B$$

14. A regular hexagon is immersed in water with a side in the surface Find the depth of its c p

15. A cubical box, filled with water, has a close-fitting heavy lid which can turn round smooth hinges attached to one edge Through what angles must the box be severally tilted about the different edges of its base so that the water may just begin to escape ?

16. There are n liquids of densities ϱ, 2ϱ, $n\varrho$ placed one above another in strata of equal depths b. A rectangle, whose sides are a and nb, is immersed vertically with a side of length a in the topmost surface. Find the position of the c p

17. An ellipse is immersed in a liquid with its plane vertical and centre at a depth h ($>$ the semi-major axis) If the ellipse be turned round its centre, find the locus of the c p (i) relative to the area, (ii) in space

18. A square lamina $ABCD$, of side a, is placed with the corner A fixed at a depth b ($> a\sqrt{2}$) below the surface, the plane of the lamina being vertical If the lamina be turned in its plane about A, find the locus of the c p in the plane

CHAPTER V

FLUID THRUSTS ON CURVED SURFACES

35. In this chapter we shall be considering curved surfaces in contact with fluids at rest under gravity.

Suppose a surface is in contact with a fluid This surface can be divided into elementary portions each of which may be regarded as plane and fluid pressure on which will be normal to that element In this manner we shall get a system of fluid pressures, and our object is to determine their resultant It is clear that this system of forces, being nonplanar, cannot in general be reduced to a single resultant force It is proved in higher statics that such a system is equivalent to a single force *together with* a single couple (whose plane does not in general contain the former force) We propose to find the resultant pressure in the form of three (or two) forces acting along lines which may not necessarily be coplanar

Fluid pressure on every element is resolved firstly into vertical and horizontal components (acting on the element). Now, all vertical components being parallel, form a system of parallel forces which has a definite resultant R acting along a determinate line This resultant is called the *resultant vertical thrust* of the fluid on the surface

Next consider the horizontal components They are not parallel to each other since horizontal lines can be drawn in infinite directions Hence we resolve them further along two conveniently chosen directions, both horizontal, say Ox and Oy The components parallel to Ox will form a system of parallel forces, and those parallel to Oy will constitute another The resultants X and Y of these two

systems can be found in magnitude and direction, they are called the *horizontal fluid thrusts parallel to Ox, Oy* respectively.

The three resultants X, Y and R will together give us the resultant of the fluid pressures on the given surface. As has been said before, they will in general act along non-planar lines. If in a particular case they are concurrent we can determine their resultant, which we may call the *resultant fluid thrust* on the surface If X and Y intersect, we can compound them, and shall obtain the horizontal fluid thrust

36. Resultant Vertical Thrust. *If a surface (which is such that no vertical line cuts it in more than one point) be in contact with fluid at rest under gravity, the resultant vertical thrust is equal to the weight of the superincumbent fluid and acts through the centre of gravity of the superincumbent fluid*

Let ABC be such a surface and we are required to determine the vertical fluid thrust on the upper side of this surface From every point on its boundary draw vertical lines to meet the free surface (where the pressure is taken to be zero) along a closed curve $A'B'C'$ Divide the surface into small elements such as a at P, let a' at Q be its projection on the free surface Let θ be the angle between the vertical and the normal, PN, to the surface at P. Then $a' = a \cos \theta$.

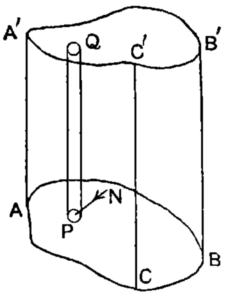


FIG 30

Now pressure on the element at $P = g\varrho a \ PQ$, along the normal if the fluid be homogeneous.

$\therefore$ its vertical component

$$= g\varrho a \,.\, PQ \cos \theta = g\varrho a' \,.\, PQ$$
$$= \text{weight of the cylinder } PQ \text{ of fluid.}$$

That is, the vertical component of the pressure is the same as the weight of the column PQ and they act along the same line. Therefore the resultant of the system of vertical thrusts will be identical with that of the system of weights of columns like PQ, that is, the resultant vertical thrust is equal to the weight of the superincumbent fluid (in magnitude and line of action) Hence the proposition.

Note 1. In the above proof the fluid is assumed to be homogeneous When it is not so and the fluid consists of layers of different fluids which do not mix, the pressure on the element at P is, by Art 10,

$$g(\varrho_1 \; QQ_1 + \varrho_2 \; Q_1Q_2 + \dots + \varrho_n \; Q_nP)\, a = wa, \text{ say}$$

$\therefore$ its vertical component $= wa \cos\theta = wa' =$ weight of the cylinder PQ of fluids, as before

Note 2. If the atmospheric pressure is to be taken into account, we take the plane $A'B'$ to be the effective surface (Art 11) of the liquid or of the uppermost liquid if there are layers of several liquids The pressure at $A'B'$ is then zero and the above theorem is applicable

Note 3. *Second proof* As in Art 22, we consider the equilibrium of the portion of the fluid enclosed between the surface AB, its projection $A'B'$ on the free surface and the cylindrical surface generated by the vertical lines, viz of the superincumbent fluid The two vertical forces, the weight of this volume of the fluid and the resultant vertical thrust on the surface must balance each other, whence the desired result

This proof is not, however, always suitable, as in the case where there is no liquid above the surface, which is considered in the next article

37. Suppose a vessel as shown in the figure is filled with a liquid and it is required to find the vertical fluid thrust

on the portion of the inner surface between the horizontal planes AD and BC. Obviously, the fluid thrust on every element and therefore the resultant vertical thrust will be in the upward direction.

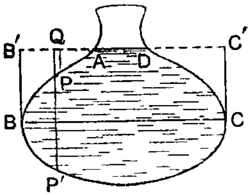


FIG. 31

In this case there is actually no superincumbent liquid, and so it may seem that the last theorem could not be applied. But we can get over this difficulty by assuming the presence of liquid (of the same kind) all round the vessel, the thickness of the material of the vessel being neglected. If the liquid inside be not homogeneous, the external liquid is assumed to be likewise, so that the density in the same horizontal level (inside and outside) is the same. Now by drawing verticals from every point of the boundary BC to meet the plane of AD, we get the superincumbent liquid as lying between the plane $B'C'$, the surface of the vessel and the cylindrical surface generated by the vertical lines.

To justify this procedure we remark that the pressure on an element a at P of the surface will be proportional to its depth below the free surface (AD), and so is equal to $g\varrho a \cdot PQ$. Therefore its vertical component $= g\varrho \cdot PQ \cdot a'$ $=$ wt. of the column PQ of the liquid, as in the previous article. Thus the rule

To find the superincumbent fluid, draw verticals from every point of the perimeter of the given surface to meet the plane of the free or the effective surface (where the pressure is zero). Then the volume of the fluid (existent or not) enclosed between the plane of the free surface, the given surface and the cylindrical surface is the required volume.

38. When the surface is not like the one considered in Art. 36, we divide the whole surface into two or more parts

each of which satisfy the condition stipulated in that article and apply the theorem to each portion. Thence the resultant vertical thrust can be easily determined

For example, let the flask (Fig 31) be filled with water up to the level AD and let the vertical fluid thrust on its surface be required In this case a vertical line like $P'Q$ cuts the surface in two points P' and P, so we cannot at once utilise the theorem of Art 36 Let us divide the surface by the plane BC, we get two portions to each of which the theorem is applicable

On the upper part, the vertical thrust *acts upwards* and its magnitude = wt of the superincumbent liquid which would lie between the portion of the surface of the vessel, the plane $B'C'$ and the cylindrical surface

On the lower part, the vertical thrust *acts downwards* and its magnitude = wt. of the superincumbent liquid lying between the lower portion of the vessel, the plane $B'C'$ and the cylindrical surface

$\therefore$ the resultant vertical thrust on the whole surface is the resultant of the above two pressures, it obviously *acts downwards* and its magnitude = wt of the latter volume − wt of the former volume of the liquid

= wt. of the liquid contained in the vessel

The line of action of this thrust passes through the c g of the contained liquid because, by statical principles, the above process is equivalent to the removal of the former volume from the latter.

39. The above is a particular case of a more general result, which is often helpful in solving problems· *If liquid be contained in a vessel and if the pressure at its free surface be zero, then the resultant fluid pressure on the vessel is equal to the weight of the liquid in the vessel and acts vertically downwards through the centre of gravity of this liquid*

The proof is as follows: Consider the equilibrium of the mass of the liquid within the vessel. There are two forces acting on it, viz. its weight and the pressure of the enclosing vessel Hence these two are equal and opposite; but the pressure of the vessel on the liquid is equal and opposite to the pressure of the liquid on the vessel (Newton's third law). Therefore the fluid pressure on the vessel is equivalent to the weight of the contained liquid, whence the theorem.

A similar result is enunciated and proved in the next article.

40. *The resultant fluid thrust on a closed surface entirely lying within the mass of a fluid at rest under gravity, is vertical, equal in magnitude to the weight of the displaced fluid and acts upwards through the centre of gravity of the displaced fluid.* (*Principle of Archimedes*)

Let $ACBD$ denote the closed surface and $A'B'$ the free surface of the fluid The fluid pressure on $ACBD$ depends wholly on the form and the position of this surface and on the fluids surrounding it. Hence it would not in any way be affected if we suppose that the volume enclosed is occupied by the same kind of fluid as outside (so that the density at the same horizontal level is the same as that of the external fluid).

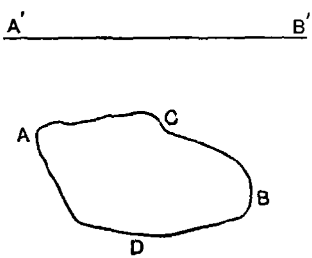


FIG. 32

The internal and the external fluids will then form a continuous mass at rest under gravity Therefore the volume of the internal fluid is also at rest under the influence of its weight and the fluid pressure of the surrounding fluid on its envelope $ACBD$. Hence these must balance each other

Thus the resultant fluid pressure on the closed surface $ACBD$ is vertically upwards through the c.g. of the volume $ACBD$ and its magnitude is equal to the weight of the volume $ACBD$ of the fluid

Note. It is not necessary for the truth of this theorem that the fluid should be homogeneous The density of the fluid may vary from one horizontal layer to another. It may, for example, consist of a number of different fluids which do not mix with one another and which rest in horizontal strata one above another. But the displaced fluid must be taken (as has been already pointed out in the proof) to be of the same density at any point as that of the surrounding fluid at the same horizontal level This should be kept in mind when calculating the weight and the position of the c g of the displaced fluid.

41. We can prove, as in Art 38, that the resultant vertical thrust on the closed surface $ACBD$ (Fig 32) is upwards and equal to the weight of the displaced fluid ; but it would not indicate that this thrust denotes the whole effect of the surrounding fluid on the surface unless it is shown that the horizontal fluid thrusts (Art. 35) are zero (see Art 48)

42. *If a solid body be partly in contact with and partly above the free surface of a mass of fluid at rest under gravity, then the resultant fluid thrust on it will be equal in magnitude to the weight of the displaced fluid acting vertically upwards through the centre of gravity of the displaced fluid*

Replace the portion of the solid below the free surface of the fluid, as in Art. 40, by similar fluid and consider the equilibrium of this new volume of the fluid introduced Assuming the pressure on the free surface to be zero, we get two forces acting on this volume, viz. the weight of the displaced fluid and the fluid pressure on the bounding

surface, these must therefore balance each other (cf. Art. 40).

43. If the free surface be the surface of separation of a liquid and the atmosphere, the pressure on it is not zero. In this case we have to replace the whole body by the corresponding volumes of the air and the liquid displaced by the body and consider the equilibrium of this combined volume of air and liquid. In fact, this case falls under the category of Art 40, since the body is wholly immersed in the fluid (of two kinds) So, as before, the resultant fluid thrust = weight of the liquid displaced + weight of the air displaced

As the latter is insignificant as compared with the former, it is usually neglected in practice, and the resultant fluid thrust is taken simply to be equal to the weight of the displaced liquid acting vertically upwards through the c g of the liquid displaced

44. It should be noted that the above theorem does not hold if the solid body, whether wholly or partially immersed in a liquid, has a *finite* portion of its surface coincident with the surface of the vessel containing the liquid Because, if we replace the solid by the same volume of the liquid and consider the equilibrium of this volume, we shall see that besides the two forces of Art 40 there are pressures of the sides of the vessels where the two surfaces are coincident, and so we cannot come to the same conclusion as in the previous article

The resultant fluid thrust of Arts. 40 and 42 is called the *force of buoyancy* and the centre of gravity of the displaced fluid (through which the former acts) is called the *centre of buoyancy*.

45. *Ex* 1 A hollow sphere of radius a is just filled with water, find the resultant vertical thrusts on the two

portions of the surface divided by a plane at depth c below the centre

Let CD denote the plane; so that $OH = c$. The sphere being just filled, the free surface (where the pressure is zero) is a horizontal plane through A, the highest point of the sphere, since the pressure obviously is least here, and *fluid pressure can never be negative* Describe a cylinder on CD as base and altitude AH.

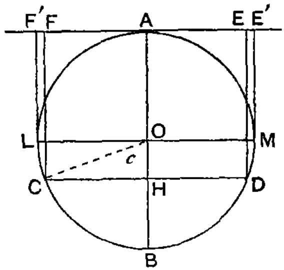


FIG 33

Then the *downward* thrust on the lower portion CBD

$$= \text{wt of the superincumbent liquid } FCBDE$$

$$= \text{wt of cyl } FCDE + \text{wt. of segment } CBD$$

$$= w\left[\pi(a+c)(a^2-c^2) + \frac{\pi}{3}(a-c)^2(2a+c)\right]$$

$$= \frac{\pi w}{3}(a-c)(5a^2+5ac+2c^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The upper portion CAD consists of two parts, the upper hemisphere LAM and the zone $LCDM$; the vertical thrust on the former is *upwards* and

$$= \text{wt of the superincumbent liquid } F'LAME'$$

$$= \text{wt of cyl } F'LME' - \text{wt of hemisphere } LAM$$

$$= w[\pi a^3 - \tfrac{2}{3}\pi a^3] = \tfrac{1}{3} w\pi a^3$$

The vertical thrust on the zone is downwards and

$$= \text{wt of the superincumbent liquid}$$

$$= \text{wt of segment } LCDM + \text{wt of cyl } F'LME' - \text{wt of cyl } FCDE$$

$$= w[\pi(a^2c - \tfrac{1}{3}c^3) + \pi a^3 - \pi(a+c)(a^2-c^2)]$$

$$= \tfrac{1}{3}\pi w(3ac^2 + 2c^3)$$

Hence the vertical thrust on the upper segment CAD is upwards and

$$=\tfrac{1}{3}\pi w(a^3-3ac^2-2c^3). \quad \text{.................}(2)$$

It may be noted that the resultant of (1) and (2), *i e.* the resultant vertical thrust on the whole spherical surface is downwards and $=\frac{4}{3}\pi wa^3$ or the weight of the contained water (cf Art. 39)

Ex. 2. A hollow closed hemispherical vessel is filled with liquid (of specific weight w) and held with its plane base downwards and inclined at an angle α to the horizontal Find the vertical and the horizontal fluid thrusts on the spherical surface

The following is an easier way of solving this problem

We know, by Art 39, that the resultant fluid thrust on the whole vessel (both on the curved surface ADB and the plane base AB) = wt of the contained liquid

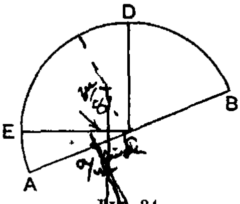


Fig. 34

$$=\tfrac{2}{3}\pi wa^3, \text{ acting downwards} \quad ..(1)$$

Now D, being the topmost point of the liquid, is on the free surface, and the centre C of the base is at depth DC or a below the free surface.

$\therefore$ pressure on the base $=\pi a^2 \cdot aw=\pi wa^3$

This pressure has vertical component $=\pi wa^3\cos\alpha$ (downwards) and horizontal component $\pi wa^3\sin\alpha$ along EC

Hence the pressure on the curved surface must have vertical component

$$=\pi wa^3(\tfrac{2}{3}-\cos\alpha)$$

downwards and $\pi wa^3\sin\alpha$ along CE, in order that these and the component pressures on the base may give the resultant (1).

Thus the resultant fluid thrust on the curved surface is of magnitude $\frac{1}{3}\pi wa^3\sqrt{13-12\cos\alpha}$, and acts in a downward direction inclined at an angle $\tan^{-1}\left(\dfrac{2-3\cos\alpha}{3\sin\alpha}\right)$ to the

horizontal The line of action of the resultant thrust can also be determined; for the weight of the liquid or the resultant (1) acts vertically through a point, on the central radius, distant $\frac{3a}{8}$ from the centre, and the pressure on the base acts normally to the base through a point $\frac{a}{4}\sin\alpha$ from C (on the line CA, see Art 34) Where these two forces intersect must give a point on the line of action of the required thrust

EXAMPLES. 5.

1. A vessel in the shape of a closed hollow cone is just filled with water and kept with its base downwards and horizontal. Show that the resultant vertical thrust on the curved surface is $\frac{2}{3}$ that of the base.

2. A right circular cylinder is just immersed in water with its axis horizontal Compare the vertical thrusts on the two parts of the curved surface into which it is divided by the horizontal plane through the axis

3 A solid right cone, height h and radius a (of the base), is completely immersed in a liquid with its axis horizontal and at depth b Compare the vertical fluid thrusts on the two halves into which the curved surface is divided by a horizontal plane through the axis

4. A hemispherical bowl of mass (72π) grammes is placed with its rim downwards on a horizontal plane which it fits closely, water is poured into the bowl through a hole at the top of the curved surface Find the height of water in the bowl when it is on the point of being lifted and water begins to escape The radius of the bowl is 13 cm.

5. A right cone is filled with water, it is then closed and the whole suspended by a point on the rim of its circular base Neglecting the weight of the conical envelope, find the resultant vertical and horizontal thrusts on the curved surface.

6. A solid hemisphere is placed with its base above the curved surface and inclined at an angle a to the surface of the liquid, in which it is just totally immersed If the resultant thrust on the curved surface be equal to twice the weight of the liquid displaced, find a.

46. Horizontal Fluid Thrust. *The horizontal fluid thrust on a given surface in contact with fluid in any given direction is equal, in magnitude and line of action, to the fluid pressure on the projection of the surface on a vertical plane perpendicular to the given direction, provided any line drawn in this direction does not intersect the surface in more than one point*

Let ABC be *such* a surface and AK the given direction. Draw a vertical plane perpendicular to AK and let lines from every point of the boundary of the given surface meet this plane in a closed curve $A'B'C'$ Then $A'B'C'$ will be the projection of the surface on this plane

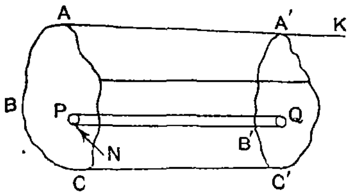


FIG 35

First proof Divide the given surface into small elements, like a at P, let a' at Q be its projection on the plane $A'B'C'$ If a and a' are sufficiently small, then pressure at any point on these two elements are equal (Art 8), let p denote this pressure

Pressure on the element at $P = pa$, acting along the normal NP

$\therefore$ its horizontal component in the direction KA $= pa\cos\theta = pa' =$ pressure on the plane-element at Q, where θ is the angle between PN and AK

Hence the horizontal components of fluid pressure on all the elements of the given surface are the same as the fluid pressures on the corresponding elements of the projection $A'B'C'$

$\therefore$ the resultant horizontal thrust on the given surface in this direction is the same as the resultant fluid thrust on the plane area $A'B'C'$, which proves the proposition.

47. *Second proof* Consider (Fig. 35) the equilibrium of the mass of fluid enclosed between the given surface ABC, its projection $A'B'C'$ and the cylindrical surface generated by lines parallel to AK This volume of the fluid, being a part of the whole at rest, is in equilibrium Therefore the components of the forces (acting on this volume) in the direction AK must also be in equilibrium These are, (1) the component of the fluid thrust on ABC in this direction, and (2) the fluid pressure on the plane area $A'B'C'$ Therefore these two fluid pressures must be equal in magnitude and act (in opposite directions) along the same line

This theorem gives us the line of action of the component since we can find out the position of the c p of the area $A'B'C'$ through which, clearly, this must pass

The direction of the component is from the fluid to the side of the surface under consideration For example, if we consider the right-hand side of the surface, the corresponding component will act from right to left

48. If the surface be not such as stipulated in the previous articles, we divide the given surface into two or more parts in each of which the condition is satisfied. We then apply the preceding theorem to obtain the horizontal component on each part separately and compound them by the rule of statics, the resulting pressure is the horizontal fluid thrust on the whole of the given surface in this direction

One case is worthy to be noticed If the surface can be divided into two parts only, the projections of which on the plane perpendicular to AK are identical, the two components of the fluid thrust, one on each part, are equal in magnitude and act along the same line If, moreover, the whole surface is a continuous one, as is usually the case, the above two components will clearly be in opposite directions, and so they cancel each other Thus there is no

horizontal component of the fluid thrust on the given surface in this direction The case discussed in Art. 41 is of such a type.

As another example we may consider the curved surface of a right circular cone immersed in a liquid with its axis vertical Choose a horizontal direction, say AB Take the section OCD of the cone, perpendicular to AB Any line drawn parallel to AB will clearly cut the conical surface in two points, but it will cut each of the two halves of the surface (by the plane OCD) in one point only Thus the condition of Art 46 is not satisfied for the whole surface, although it holds for each of the two halves The projection of each portion on a vertical plane perpendicular to AB will be identical with the section OCD; therefore, by Art 46, the horizontal components of fluid pressure on the two portions are equal in magnitude and have the same line of action Moreover, the pressure on the right half acts from right to left, that on the left half acts from left to right, *i e* they are in opposite directions Hence, on the whole surface, the horizontal fluid pressure in the direction AB is zero

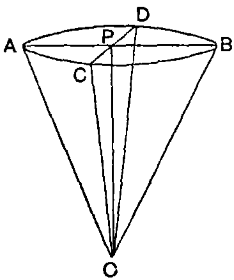


FIG 36.

It can similarly be shown that there is no horizontal component in any other direction Thus there is no horizontal fluid thrust on this solid cone The fluid thrust is therefore wholly vertical

It is easy to see that if the given surface has a vertical plane of symmetry, there will be no horizontal fluid thrust on the surface in the direction perpendicular to this plane

49. As has been stated in the beginning of this chapter, the resultant horizontal fluid thrust is obtained by com-

pounding, wherever possible, the horizontal components in two convenient directions at right angles

Ex 1. A vessel in the form of a cone with its vertex downwards and axis vertical is filled with water Find the fluid thrust on one of the halves into which the surface of the vessel is divided by a plane through the axis

Let AOB (Fig 36) be the vessel, and let it be required to find the fluid thrust on the half $OCBD$

The vertical thrust = wt. of superincumbent water

$$= \text{wt of vol } OCBD \text{ of water}$$

$$= \tfrac{1}{6} w\pi a^2 h \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Horizontal thrust in the direction AB

$$= \text{pressure on the projection } OCD$$

$$= w \,.\, ah \,.\, \tfrac{1}{3}h = \tfrac{1}{3}wah^2 \quad \ldots\ldots\ldots\ldots (2)$$

Horizontal thrust in the direction $CD = 0$,

since the plane OPB is a plane of symmetry of the semi-conical surface (Art 48) The force (1) acts through the c g of the semi-cone which is in the plane OPB ; its line of action is parallel to OP and at a distance of $\dfrac{a}{\pi}$ from it The force (2) acts through the c p. of the triangle OCD, its line of action is in the plane OPB and perpendicular to OP at its middle point These two intersect at a point X whose position is thus determined The magnitude of the resultant fluid thrust $= \tfrac{1}{6}wah\sqrt{\pi^2a^2+4h^2}$ and acts through X, in a direction making an angle $\tan^{-1}\left(\dfrac{2h}{\pi a}\right)$ with the downward vertical

Ex 2 Two closely fitting hemispheres made of a material of uniform thickness are hinged together at a point on their rims, and are suspended from the hinge The rims are greased so that the whole forms a water-tight vessel. It is being filled with a liquid through a small hole near the

hinge. What must the weight of the vessel be so that no liquid may escape even when the vessel is filled ?

Let A be the hinge and the two hemispheres be ACB and ADB The liquid will escape if the two parts begin to separate. We have, therefore, to find the condition so that this may not happen

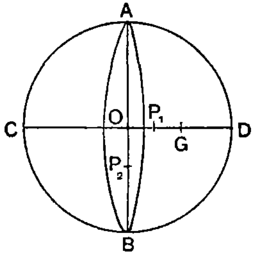


FIG 37

Consider the forces acting on one hemisphere, they are. (1) the action of the hinge at A, (2) its weight W' acting at G where $OG=\frac{a}{2}$, (3) the vertical fluid thrust equal to the weight of the hemispherical volume of the liquid, say W, acting downwards through P_1 where $OP_1=\frac{3}{8}a$, (4) the horizontal fluid thrust equal to the pressure on the section AB through its c p, P_2, where $OP_2=\frac{1}{4}a$ [Art 34], the magnitude of this last $=\pi a^3 w=\frac{3}{2}W$.

Taking moment about the hinge, we see that the force (1) can be neglected, that the forces (2) and (3) tend to keep the hemispheres together and that the force (4) tries to separate them Hence the required condition would be that

the sum of moments of (2) and (3) $>$ or at least $=$ the moment of (4)

Substituting, $\quad W'\cdot\frac{a}{2}+W\cdot\frac{3}{8}a > \text{or} = \frac{3}{2}W\left(a+\frac{a}{4}\right)$,

whence $\quad W' >$ or $=3W$, or $2W' >$ or $=3\,.\,(2W)$,

i e. wt of the shell $>$ or $=3$(wt of the contained liquid)

EXAMPLES. 6.

1. A vessel in the form of an octant of a sphere (radius a) with plane faces is filled with water and placed with one diametral plane horizontal and uppermost. Find the resultant fluid thrust on the curved surface.

2. A right circular cylinder is just immersed in water with its axis horizontal. Compare the horizontal fluid thrusts on the four parts of the curved surface made by the horizontal and vertical planes through the axis.

3. A right cone whose base is an ellipse of semi-axes a, b, is divided into two equal parts by a plane through its axis and the major axis of the base One part is then removed and just immersed in a liquid with its vertex upwards and the axis of the cone vertical. Find the horizontal thrust on one-half of its curved surface made by a plane perpendicular to the vertical face of the solid.

4. A hemispherical bowl is filled with water, find the horizontal fluid pressure on one-half of the surface divided by a vertical diametral plane and show that it is $\frac{1}{\pi}$ of the magnitude of the resultant fluid thrust on the whole surface

5. A cone, vertex upwards, is immersed in water with the centre of its base at a distance of $\frac{5}{8}$ths of its altitude below the surface and its axis inclined at 60° to the vertical A paraboloid of the same base and altitude is also immersed with its vertex upwards, the centre of the base at the same distance below the surface as that of the cone, and with its axis inclined at the same angle to the vertical Prove that the resultant fluid pressures on the curved surfaces of the two solids are equal in magnitude and find their common value

6. A hollow ellipsoidal shell with semi-axes a, b, c is held with the c-axis horizontal and the a-axis at an angle α with the vertical Liquid of density ϱ is poured in through a small hole at the top of the shell till it is filled Find the vertical and horizontal thrusts on the two halves of the shell made by the vertical plane through the a- and the b-axes Density of water is ϱ_0 and the height of water barometer is H.

7. A hemispherical bowl is half-filled with water and the other half is then filled with oil of sp. gr. $\frac{1}{2}$. Find the horizontal and vertical thrusts on one half of the surface divided by a plane through the vertical radius.

8. A hemispherical bowl is filled with water; show that the horizontal and vertical fluid thrusts on a portion of the curved surface of the bowl intercepted between two planes through the vertical radius are $\frac{2}{3}wa^3 \sin \alpha$ and $\frac{2}{3}wa^3\alpha$ respectively, where 2α is the angle between the planes

9. A solid sphere of density ϱ is placed at the bottom of a vessel, which is horizontal, and a liquid of density $\sigma(<\varrho)$ is

poured in so as just to cover up the sphere. The sphere is then cut along a plane through the vertical diameter. Prove that the two parts will not separate unless $\varrho > 4\sigma$

10. If one side of a solid completely immersed in a mass of fluid at rest under gravity be a plane, show how the calculation of the resultant fluid thrust upon the curved surface is facilitated

11. A solid cone is held with its axis in the surface of water. Find the resultant liquid pressure on the immersed portion of the curved surface in magnitude and direction.

12. A hollow closed cone made of material of density ϱ' and of uniform thickness, can rest in all positions, wholly submerged, in a liquid of density ϱ Prove that the semi-vertical angle of the cone must be $\sin^{-1}\frac{1}{3}$ and that the thickness of the shell is approximately equal to $\frac{h}{12}\frac{\varrho}{\varrho'}$, where h is the height of the cone.

13. A solid cone whose vertical angle is 60°, is just immersed in water with a generating line in the surface Prove that the resultant thrust on the curved surface of the cone is equal to $\frac{\sqrt{7}}{2}$ times the weight of the water displaced by the cone and find the inclination of its line of action to the vertical

CHAPTER VI

EQUILIBRIUM OF FLOATING BODIES

50. It has been proved in the last chapter (Arts 40-44) that when a solid body is wholly or partly immersed in fluid it experiences a thrust, vertically upwards, whose magnitude is equal to the weight of the displaced fluid and whose line of action passes through the centre of gravity of the displaced fluid Therefore, whenever we have to consider the equilibrium of a solid in contact with fluids we have to introduce this force along with the other external forces acting on the body Then we apply the usual principles of statics and get three equations (or less) by the help of which the problem can be solved.

51. As the simplest case, let us consider the equilibrium of a floating body under the action of gravity alone Let $ABCD$ be the body of which the portion ABC is under the liquid, let the c g of the whole body be at G and that of the submerged part (or the centre of buoyancy) be at H. Also suppose that the weight of the body is W and that of the displaced liquid is W' Let the effect of the pressure of air be not taken into consideration

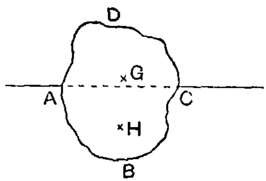


FIG 38

The solid is in equilibrium under the influence of two forces only, its weight W acting downwards from G and the fluid thrust W' acting upwards through H Hence for the equilibrium position, $W = W'$, and they must act along the same vertical line, *i e* GH must be vertical.

Thus we get the well-known principles . *that a floating body must displace fluid of such volume (or volumes) that its weight would be equal to that of the body, and that the position of the body should be such that the centres of gravity of the whole body and of the submerged part are in the same vertical line*

52. If the pressure of atmosphere be included, we see that the fluid thrust on the body consists of two parts, viz the weight of the volume ABC of the liquid and the weight of the volume ADC of air, each acting upwards through the centre of gravity of the respective volume Hence from the consideration of equilibrium we see that the weight of the body is the resultant of the weights of the displaced fluids, so that its magnitude = the sum of the latter

Similar remarks will apply if the body displaces volumes of two or more fluids which do not mix, or if the fluid displaced be heterogeneous, the density at any point being a function of the depth of the point.

53. *Ex* 1 A cylinder of wood (sp gr $\frac{3}{4}$) of length h, floats with its axis vertical in water and oil (sp gr $\frac{1}{2}$), the length of the solid in contact with ~~oil being~~ $a\,(<\frac{1}{2}h)$ Find how much of the wood is above the liquids , also find to what additional depth must oil be added so as just to cover the cylinder

Let x be the length of the solid *above* the liquids , since a length a is in contact with oil, the remainder $h-a-x$ is in water

Weight of the solid $=\frac{3}{4}whA$, where A is the area of its cross-section and w is the weight of unit volume of water Weight of oil displaced $=\frac{1}{2}waA$, and that of water displaced $=w(h-a-x)A$.

$$\therefore\ \tfrac{3}{4}whA=\tfrac{1}{2}waA+w(h-a-x)A,$$

or
$$\tfrac{3}{4}h=\tfrac{1}{2}a+h-a-x,$$

whence
$$x=\tfrac{1}{4}h-\tfrac{1}{2}a$$

Since a is given to be less than $\frac{1}{2}h$, the value of x is positive If, however, a were greater than $\frac{1}{2}h$, the value of x would have been negative, showing thereby that the hypothesis as well as the above equations were wrong. In that case no part of the cylinder would be above the surface of the oil, its length l in contact with oil being given by

$$\tfrac{3}{4}h = \tfrac{1}{2}l + (h - l),$$

whence $l = \frac{1}{2}h$

Next, let y be the additional depth of oil which is necessary just to cover up the cylinder, then the new depth of oil, and so the length of the solid in contact with oil $= a + y$ $\therefore$ its length in water $= h - a - y$

Hence, as before,

$$\tfrac{3}{4}whA = \tfrac{1}{2}w(a + y)A + w(h - a - y)A,$$

or $$\tfrac{3}{4}h = \tfrac{1}{2}(a + y) + h - a - y,$$

whence $$y = \tfrac{1}{2}h - a,$$

which is positive since it is assumed that $h > 2a$

Ex 2 A thin uniform rod of weight W has a particle of weight w attached to one end It is floating, in an inclined position, in water with this end immersed Prove that the length of the rod above water is $\dfrac{w}{w + W}$ times its whole length and that the specific gravity of the rod is $\dfrac{W^2}{(w + W)^2}$

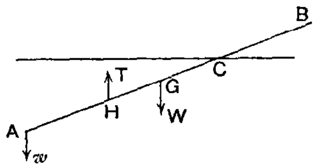


FIG 39

Let ACB be the rod, CB being the part above water. Let G, H be the middle points of AB, AC respectively The forces on the rod are its weight W at G, the weight w of the particle at A and the fluid thrust T (which is equal to the weight of length AC of water) at

H. Therefore, since T would be the reversed resultant of w and W,

$$\frac{w}{W}=\frac{HG}{AH} \text{ or } \frac{w}{w+W}=\frac{HG}{AG}=\frac{2HG}{2AG}=\frac{2(AG-AH)}{2AG}$$

$$=\frac{AB-AC}{AB}=\frac{CB}{AB}$$

$$\therefore\ CB=\frac{w}{w+W}\cdot AB.$$

Again, $T=w+W$, also $=$ weight of length AC of water $=\frac{\text{weight of length } AC \text{ of the rod}}{\sigma}$ where σ is the sp gr of the rod

$$\therefore\ T=\frac{AC}{AB\,.\,\sigma}\ W=\frac{W}{w+W}\ \frac{W}{\sigma}=W+w,$$

whence
$$\sigma=\frac{W^2}{(w+W)^2}$$

Ex 3 A uniform isosceles triangular lamina (sp gr σ) floats in water with its plane vertical, its vertical angle (which is equal to 2α) immersed and the base wholly above the water Prove that in the position of equilibrium in which the base is not horizontal, the sum of the lengths of the immersed portions of the two sides is $2a\cos^2\alpha$ where a is one of the equal sides, and that σ is less than $\cos^4\alpha$ as well as $\cos 2\alpha$

Obviously one position of equilibrium is the symmetrical one, the base being horizontal, since in this case both the conditions of equilibrium (Art. 51) are satisfied But we are required to discuss the other positions of equilibrium, if there are any. Assuming the existence of such a position we proceed to solve the problem. If, on this assumption we arrive at some result which is absurd or incompatible with our hypothesis we must at once reject the possibility of the existence of such positions, otherwise not The following figure is drawn with C lower than B, clearly

there can be a similar position with B lower than C. Thus there may be three positions of equilibrium or only one (symmetrical)

Let ED be the section by the surface of water. We have to prove, firstly, that $AD + AE = 2a \cos^2\alpha$

Let AK, AL be the medians of the triangles ABC, ADE and G, H their centroids Then the weight of the lamina acts through G and the fluid thrust through H Therefore for equilibrium position, GH must be vertical Since KL is parallel to GH, it follows that KL is vertical, *i e* KL is perpendicular to ED (which is clearly horizontal, being the section of the water-surface) Also L is the middle point of ED; $\therefore$ $KE = KD$

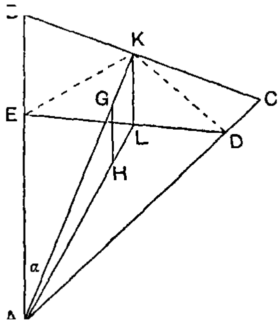


FIG 40.

$\therefore$ equating the values of KE^2 and KD^2 obtained from the triangles AKE and AKD,

$$AE^2 + AK^2 - 2AE \cdot AK \cos a$$
$$= AD^2 + AK^2 - 2AD \cdot AK \cos a,$$

or
$$AD^2 - AE^2 = 2AK(AD - AE) \cos a$$

$\therefore$ either $AD = AE$ which we shall for the present reject since it will give the symmetrical position, or

$$AD + AE = 2AK \cos a = 2a \cos^2 a. \quad \ldots\ldots\ldots(1)$$

Next, we have for equilibrium,

wt of $\triangle ABC$ of solid = wt of $\triangle AED$ of water,

or
$$\sigma w \cdot \tfrac{1}{2}AB \cdot AC \sin 2a = w \cdot \tfrac{1}{2}AD \cdot AE \sin 2a,$$

whence
$$\sigma = \frac{AD \cdot AE}{a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

But we know that $4ab < (a+b)^2$ when a and b are unequal.

$$\therefore \ \sigma < \frac{(AD+AE)^2}{4a^2}, \text{ or } < \cos^4 \alpha, \text{ from (1).}$$

Lastly, to find the other limit of σ we remark that the greater σ is, the greater will be the portion in water, viz. the triangle ADE But by hypothesis, the base BC must always be wholly above water. Therefore the limiting case would occur when DE passes through the lower corner C. In that case $AD = AC = a$, and by (1)

$$AE = 2a \cos^2 \alpha - a = a \cos 2\alpha$$

$$\therefore \text{ by (2), } \sigma \text{ would be equal to } \frac{a \cdot a \cos 2\alpha}{a^2}, \text{ or to } \cos 2\alpha$$

$\therefore$ $\cos 2\alpha$ is the greatest value of σ in order that the hypothesis may not be violated.

54. Let us next turn to the cases which involve other forces besides those of gravity The method to be followed can best be illustrated by solving some examples

✓*Ex* 1 A rod of length $2l$ and sp gr ϱ is hinged at a point at a height h above the surface of a liquid of sp gr $n\varrho$ (n being >1) The rod remains in equilibrium in an inclined position, partly immersed in the liquid Find the inclination of the rod to the vertical, and the length immersed

Let the rod be ACB of which the length CB is in the liquid. $AN = h$, $AB = 2l$, let $CB = 2x$, the $\angle NAC = \theta$,

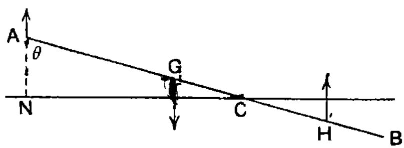


FIG 41

H and G be the middle points of CB and AB respectively The forces on the rod are (1) its weight at G downwards,

(2) the fluid thrust equal to the weight of length CB of the liquid, acting upwards at H, and (3) the reaction of the hinge at A, which must necessarily be vertical To get rid of this unknown reaction, we take moment about A If a denotes the cross-section of the rod, we then have

$$g\rho \cdot 2la \cdot AG \sin\theta - gn\rho \cdot 2xa \cdot AH \sin\theta = 0,$$

or
$$l^2 = nx(2l - x),$$

or
$$nx^2 - 2lnx + l^2 = 0 ;$$

$$\therefore\ x = l\left[1 - \sqrt{1 - \frac{1}{n}}\right],$$

rejecting the + sign because x must be less than l Double this quantity gives the length of the immersed portion

Also, $$\cos\theta = \frac{AN}{AC} = \frac{h}{2(l - x)} = \frac{h}{2l}\sqrt{\frac{n}{n-1}}$$

Note. If $h > 2(l - x)$, i e $> 2l\sqrt{1 - \frac{1}{n}}$, the above fraction becomes greater than unity, and there would be no possible value of θ In this case the rod will be hanging in the vertical position with a part ($2l - h$ which is $< 2x$) immersed in the liquid

Ex 2 A body is weighed, by means of a spring balance, in air and in water, and its apparent weights are found to be W and W' respectively Find its true weight, that is, its weight when suspended *in vacuo*, if the sp gr of air be s

The spring balance registers the force with which the body pulls the spring of the balance, which is called the apparent weight This force is the resultant of the *other* external forces acting on the body, since the body is in equilibrium under the influence of *all* the forces When the body is suspended in air, the other forces are its true weight (W_0 say), and the fluid thrust of the air which is in

opposite direction and equal to Vsw where V is the volume of the body and w is the weight of unit volume of water.

$$\therefore\ W_0 - Vsw = \text{pull on the spring} = W. \quad \ldots(1)$$

Similarly, we can show that when the body is weighed in water,

$$W_0 - Vw = \text{pull on the spring} = W' \quad \ldots(2)$$

$\therefore$ eliminating (Vw) from (1) and (2), we get

$$W_0(1-s) = W - W's, \quad \ldots\ldots \quad (3)$$

whence W_0 or the true weight can be determined

Note. When the body is weighed by means of a common balance (instead of the spring balance) whose equal arms are OA and OB, each pan with appendages and weight-pieces (or the body) exerts a vertical downward pull at A (or B) of the arm, these pulls are equal in magnitude since the arms are taken to be equal and horizontal

Now the pull exerted on one arm by the corresponding pan with the objects placed on it = the true weights of the pan and the objects – fluid thrust on them

As the true weights of the pans are equal and as the volumes of the material of which they are made are also equal, their true weights as well as the fluid thrusts exerted on them cancel each other when the two pulls are equated Hence the equation becomes

the true weights of the objects on the first pan – fluid thrust on them
= true weights of the objects on the second pan – fluid thrust on them

The fluid thrusts may be due to air, or to a liquid, or to both, according as the case may be

Ex 3 A glass, partly filled with a liquid, is placed on one pan of a common balance and is counterpoised by suitable weight-pieces placed on the other A solid which is suspended by means of a string, is now brought and

immersed in the liquid without touching the sides of the glass, being still in suspension from the string Neglecting the effect of the atmosphere, explain why some more weights are now needed on the second pan to balance the glass of water

Let us begin by explaining how the presence of water in the glass affects the arm of the balance We might say that the pull on the arm is due to the combined weight of the vessel and water contained in it (in addition to that of the pan) , but this would not be strictly accurate The more correct mode of expression would be the following : water exerts a downwards pressure on the vessel equal to its own weight when the glass contains nothing else than water (Art 39) This pressure is communicated by the glass vessel to the surface of the pan, and in consequence to the arm of the balance in the form of a pull When the solid is introduced in the volume of the water, its level rises and consequently the downward fluid pressure on the vessel is increased The pull on the arm of balance is also increased by the same amount, and therefore more weights are required on the other pan to counterbalance this additional pull

To determine the amount of this increase we remark that the fluid-pressure on the surface of the containing vessel (in the second case) will be unaffected if the solid be replaced by water of the same volume, since the pressure at any point depends on its depth below the surface and not on the presence of the solid Thus it is evident that in this case the downward fluid thrust on the vessel will be equal to the weight of the volume of water up to the higher level, the volume occupied by the solid being included, by Art 39 Therefore the increase = the weight of a volume of water equal to that of the solid immersed.

On the other hand, the tension of the string by which the body is suspended is lessened by an equal amount, because the solid now experiences an upward fluid thrust.

EXAMPLES. 7.

1. A solid whose true weight is w_0 is weighed in water and in another liquid, and its apparent weights are found to be w_1 and w_2 Calculate the sp gr. of the liquid.

2. A balloon, of volume V, contains hydrogen whose density is to that of the air at the earth's surface as $2 \cdot 29$. If the envelope of the balloon be of weight w but of negligible volume, find with what acceleration it will ascend.

3. If the balloon in the last example can float at a certain height, obtain the density of the air at that level in terms of that of the air at the earth's surface.

4. A thin uniform rod AB, of length $2a$ and weight W, has a particle of weight W′ attached to a point D near the end A, AD being of length b The rod floats, in an inclined position, freely in water with a length AC $(=2x)$ immersed. Prove that $x = \dfrac{aW + bW'}{W + W'}$ and that the sp. gr ϱ of the material of the rod is given by $a\varrho(W + W')^2 = W(aW + bW')$.

5. A ship sailing out of the sea into a river sinks through a distance b, on unloading cargo of weight P, the ship rises through a space c. Show that the weight of the ship after unloading is $\left[\dfrac{b\sigma}{c(\sigma - \varrho)} - 1\right]P$, where σ and ϱ are the sp. gr. of sea-water and river-water respectively. The cross-section of the ship near the water-level (or water-line) may be assumed to be uniform.

6. A hollow cube whose side is a and weight $W (< \frac{1}{6}a^3)$ is floating freely in water (the weight per unit volume of which is taken as unity) with a diagonal vertical Find the depth of the lowest vertex of the cube

7. Show that if a right circular cone floats freely in a liquid with its vertex in the surface, the density of the liquid must be double of that of the solid and the axis of the cone must lie on the surface of the liquid.

8. A cylindrical vessel containing some water rests on a horizontal table, if I dip my hand in the water without touching the vessel, how is the pressure on the table affected ?

9. A bucket almost full of water is suspended from a spring balance. A solid, heavier than water and of volume V, is immersed while in suspension by means of a string held in hand, it is noticed that a volume v of water overflows from the bucket. Find the change in the reading of the spring balance.

10. A uniform rod is bent into the form of the three sides AB, BC, CD of a square A is attached to a hinge fixed on the surface of water and the frame rests in a vertical plane with AB, BC and half of CD immersed in water Obtain the sp gr of the rod

11. A thin conical shell whose height is 12 in and radius of the base 4 in, has a silver sphere of diameter 3 in. placed within it The shell is floated in glycerine (sp gr 1 3) with its axis vertical and vertex downwards. If the sp gr. of silver be 10 4 and the weight of the shell be $\frac{1}{9}$th of that of the sphere, find to what depth it will sink

12. An equilateral triangular lamina ABC of sp. gr $n^2(<1)$, is movable about a fixed hinge at A and is in equilibrium when the corner C is immersed in water and AB horizontal and above the water. If the triangle be turned about A in its vertical plane and kept in the position with the side BC immersed and horizontal, prove that the action of the hinge in this position is $\dfrac{2(1-n)}{n}$ times the weight of the lamina

13. Show that a uniform lamina in the form of a parallelogram cannot float in a liquid in a vertical plane with one side (or diagonal) horizontal unless it be rectangular (or equilateral).

14. Show that a uniform triangular lamina cannot float vertically in a liquid with one side horizontal unless it is isosceles.

15. A spherical shell of copper (sp gr. 8) floats in water with half its surface immersed Find its internal radius if the external one be 10 inches

16 An alloy of gold (sp. gr. 19) and silver (sp. gr. 10 5) floats with $\frac{47}{48}$ of its volume in mercury (sp gr. 13 6) and the remainder in a liquid of sp gr 0 8 Find the proportion of gold and silver in the alloy.

17. A cylindrical vessel the area of whose cross-section is a, is placed with its base horizontal An iron cylinder (sp. gr 7 5) of height H and cross-section β, rests with its axis vertical on the bottom of the vessel Mercury (sp. gr 13 5) is poured into the vessel up to a depth $h(<\frac{5}{9}H)$, water is then poured on the top of the mercury until the iron cylinder is covered Show that the latter would rise through a height $\left(1-\dfrac{\beta}{a}\right)(h-\cdot 52H)$, if $h > 52H$.

18. A hollow conical vessel floats in water with its axis vertical and $\frac{1}{n}$ of its axis immersed. Find to what depth must the vessel be filled with water so that it may just sink till its mouth is on the level with the surface of the water outside.

19. A right circular cylinder floats in a liquid of density ϱ with half of its axis immersed, the axis being vertical. Another liquid of half the density is poured on the top of the former to a depth equal to half the height of the cylinder. Find by how much the cylinder has risen.

20. A cylinder of sp gr 2ϱ floats with its axis vertical in contact with two liquids of sp. gr ϱ and 3ϱ respectively, the height of the cylinder being equal to the depth of the upper liquid. Prove that the pressures on its ends are as 1 . 5

21. A right circular cylinder (sp. gr. σ) floats in water with its axis vertical, one-third being above the water If s be the sp. gr. of air prove that $3\sigma = 2 + s$

22. A right circular cylinder is floating in water with its axis in the surface when the sp gr of air is $\cdot 00128$ If this sp gr increases to $\cdot 0013$, find the new volume of water displaced by the cylinder and show that its axis has risen by approximately $\cdot 000016$ of the radius of the base

23. A solid is weighed by a spring balance and its weight is found to be w If the effect of the atmosphere be taken into account, prove that its true weight is to w as 1 $1 - \frac{s}{\varrho}$ where s and ϱ are the specific gravities of the air and the solid respectively.

24. A solid is weighed by means of a common balance and weight-pieces, and its weight is to be found to be w Taking the effect of the air into consideration, calculate its true weight, given that s, ϱ and σ are the specific gravities of the air, the solid and the weight-pieces respectively

25. A uniform rod, of length $2a$ and density ϱ, is movable in a vertical plane about one end which is fixed in a liquid of density ϱ' at a depth $2h$ below its surface A liquid of smaller density σ is added on the top of the first liquid, if in the oblique position of equilibrium the rod is just covered by the lighter liquid, prove that its inclination to the vertical is

$$\cos^{-1}\left[\frac{h}{a}\left(\frac{\varrho' - \sigma}{\varrho - \sigma}\right)^{\frac{1}{2}}\right].$$

26. Two uniform straight rods of lengths $2a$, $2b$ and sp. gr. ϱ, σ respectively, are joined rigidly together to form one straight rod. If it floats freely in water in an inclined position, the rod of length $2a$ and part of the other being immersed, prove that

$$a^2\varrho(\varrho - 1) + 2ab\sigma(\varrho - 1) + b^2\sigma(\sigma - 1) = 0$$

27. Two uniform rods AB, BC (of the same cross-section and of lengths a, b respectively) are hinged together at B and the system can turn freely in a vertical plane about a fixed hinge at A, which is kept on the surface of water. The rod AB is of ivory (sp. gr. 1·9) and BC of wood (sp. gr. 0·5). Show that in the position of equilibrium wherein the rods are inclined to the vertical, the portion of BC which is above water is of length $\frac{1}{2}b - \frac{9}{20}a$, if this be positive.

28. A thin hemispherical bowl has a heavy particle fixed to its rim and floats in water with the particle just above the surface when the plane of its rim is inclined at 45° to the horizontal. Compare the weight of the bowl with the weight of the water it can hold.

29. Discuss the conditions of equilibrium of a solid body which is wholly or partly immersed in a heterogeneous fluid whose density is a given function of the depth.

30. A solid hemisphere of radius a and weight W is floating in water with its base upwards; a small weight w is placed on the base at a distance c from the centre. If the solid now floats with the base above water, find its inclination to the horizontal.

31. A solid sphere of radius a is just immersed in three liquids whose densities are as 1 : 3 : 5; the two surfaces of separation of the liquids are at depths $\frac{2}{3}a$ and $\frac{5}{3}a$ from the top. Compare the density of the sphere with that of the topmost liquid.

32. A buoy made of uniformly thick sheet metal which weighs w per unit area consists of a hemisphere of radius a surmounted by a hollow cylinder of height h which is closed at the top. Prove that the buoy will float in a liquid of density ϱ without any portion of the cylinder being immersed if $2a^2g\varrho > 3w(3a + 2h)$.

33. A candle of sp. gr. ϱ floats vertically in water; it is lighted, and the flame is observed to descend towards water with uniform velocity u. Assuming the water to be still, prove that the candle burns with the velocity $\dfrac{u}{1-\varrho}$.

34. Two equal uniform rods AB, BC made of material of sp. gr. $\frac{3}{4}$, are rigidly jointed at B They float freely in water in unsymmetrical position with B immersed and C on the surface, the plane ABC being vertical Prove that BA and BC are inclined to the horizon at angles whose tangents are 2 and $\frac{1}{2}$ respectively

35. Two equal uniform rods of sp gr 75 are hinged together and their other extremities are joined together by a weightless string of such a length that the whole forms a right-angled isosceles triangle when the string is tight. This floats in the symmetrical position with the right angle immersed in water. Compare the tension of the string with the weight of each rod.

36. Two equal uniform rods AB, BC, rigidly jointed at B at an angle $2a$, float in a liquid in an unsymmetrical position with B immersed and A and C outside the liquid Prove that the ratio of the density of the rods to that of the liquid lies between $\left[\frac{\cos^2 a}{1+\sin^2 a}\right]$ and $\frac{1}{2}\cos^2 a$

37. If the system of the previous question float in a liquid of twice their density in an unsymmetrical position with the angle B outside and the free ends immersed, prove that $\cot^2\theta = \cot^2 a - 2$, where θ is the inclination of the bisector of the angle between the rods to the horizontal

38. A solid cone is divided into two parts by a plane through the axis and the parts are joined together by a hinge at the vertex. They are put together so as to form the complete cone, and in this position the system is made to float in water with the vertex of the cone downwards and the axis vertical If it floats without separation of its parts, prove that the sp gr of the material of the cone is greater than $\sin^6 a$, a being the semi-vertical angle

39. A uniform lamina in the form of an equilateral triangle floats with its plane vertical. Show that there will be only one position of equilibrium with a vertex below and the opposite side wholly above the surface of the liquid unless the ratio of the sp gr. of the liquid to that of the lamina is greater than 2

40. A heavy parabolic lamina of uniform thickness is bounded by a double ordinate whose length is $\sqrt{7}$ times the latus rectum. It floats in water with its plane vertical and the focus on the surface, the axis being inclined to the vertical

and the bounding chord wholly outside the water. Show that the axis makes an angle 45° with the vertical and determine the sp gr. of the lamina.

41. A square board is placed with its plane vertical, in a liquid of four times its density, show that there are three different positions in which it will float with two sides wholly above the surface and determine them

42. A right circular cone is floating with its axis vertical and vertex downwards in a liquid and $\frac{1}{n}$th part of its axis is immersed A weight equal to the weight of the cone is placed at the centre of the base upon which the cone sinks till its axis is totally immersed, before rising Prove that $n^3+n^2+n=7$. [Apply the principle of energy and work, assuming the surface of the liquid to be the same throughout the motion of the cone.]

55. Let a solid body float in a liquid so as to displace a volume of the liquid whose weight is equal to the weight of the body, the centre of mass of the volume immersed is called the *centre of buoyancy* (cf. Art. 44), and the section of the solid by the surface of the liquid is called the corresponding *plane of floatation*

The above definitions do not stipulate that the solid should float in a position of equilibrium under gravity On the contrary the solid may float in any manner whatsoever provided that the liquid displaced is always equal in weight to that of the body If the liquid be homogeneous, this requires that the volume of the liquid displaced should always be the same Thus, the plane of floatation and the centre of buoyancy may also be defined as follows

Let ϱ be the density of the liquid and W the weight of the solid ; let V be defined by $g\varrho V=W$. If any plane cuts off from the solid a part whose volume is equal to V, then the section of the solid by this plane is the plane of floatation and the centre of mass of the volume V cut off is the corresponding centre of buoyancy.

It is evident that a given solid may have an indefinite number of planes of floatation and corresponding centres

of buoyancy with reference to a liquid. The locus of the centres of buoyancy is said to be the *surface of buoyancy* and the envelope of the planes of floatation the *surface of floatation.*

If the body be a thin plane lamina, and if it be made to move in its plane so as to displace the same amount of liquid (whose weight would equal that of the lamina), then the section of the plane by the surface of the liquid is known as the *line of floatation,* its envelope the *curve of floatation* and the locus of centres of buoyancy the *curve of buoyancy.*

The above three terms are sometimes used to denote the section by a plane, of the plane of floatation, the surface of floatation and the surface of buoyancy respectively.

56. The centre of buoyancy is generally denoted by the letter H and the centre of gravity of the whole body by G The line joining G to H is not always perpendicular to the corresponding plane of floatation. But if H corresponds to a position of equilibrium, that is, if H be the centre of mass of the submerged volume when the solid floats in equilibrium, GH must be vertical (Art 51), and since the plane of floatation is *always* horizontal, being the plane of the surface of the liquid, GH is normal to this plane Thus the position of equilibrium has this important property: that *the line joining the centre of gravity of the whole body to the centre of buoyancy for this position is perpendicular to the corresponding plane of floatation*

57. *The tangent plane to the surface of buoyancy at any point on it is parallel to the corresponding plane of floatation*

Any point H on the surface of buoyancy is a centre of buoyancy for some position of the floating of the solid This centre, H, has the corresponding plane of floatation, say X. It is to be proved that the tangent plane at H to the surface of buoyancy is parallel to the plane X

Let ACB denote the plane of floatation and H the corresponding centre of buoyancy, i e H is the c g of the volume ADB cut off by the plane ACB

Let $A'CB'$ denote another plane of floatation which makes a small angle with the plane ACB Then the plane $A'CB'$ may be called an adjoining plane of floatation to the first plane. We can determine the corresponding centre of buoyancy, H', by the following method

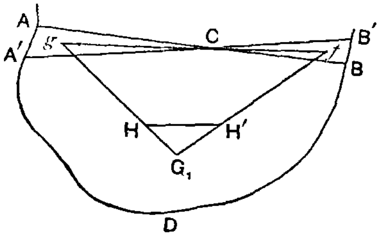


FIG 42

Let g and f denote the c g of the wedges ACA' and BCB' respectively and G_1 the c g of the volume $A'DBC$.

$$\text{vol } ADB = \text{vol } A'DBC + \text{wedge } ACA'$$

$\therefore$ wt of vol ADB is the resultant of the weights of the vols $A'DBC$ and ACA' at their respective centres of gravity, viz H, G_1 and g

$$\therefore\ gH \cdot HG_1 = \text{vol. } A'DBC \quad \text{wedge } ACA'. \ \ldots\ldots(1)$$

Since $\quad \text{vol } A'DB' = \text{vol } A'DBC + \text{wedge } BCB'$,

the c g of the vol $A'DB'$, i e H', lies on the line joining the c g of the volumes $A'DBC$ and BCB', i e on the line G_1f, such that

$$fH' \quad H'G_1 = \text{vol } A'DBC \quad \text{wedge } BCB' \ \ldots\ldots (2)$$

But wedges ACA' and BCB' are equal in volume, since, by hypothesis, the vol ADB = the vol $A'DB'$ Therefore from (1) and (2) we have

$$gH \quad HG_1 = fH' \quad H'G_1,$$

whence HH' is parallel to gf

Now let the plane $A'CB'$ be made to approach and ultimately to coincide with the plane ACB. In the limiting

position, the points g and f and therefore the line gf falls along the plane ACB Also, the line HH' becomes ultimately the tangent (at H) to the locus of the centre of buoyancy or the surface of buoyancy

Thus a tangent at H to the surface of buoyancy is parallel to a line in the plane ACB and therefore to the plane ACB itself

This would be true for all tangents Hence the tangent plane at H to the surface af buoyancy is parallel to ACB, the corresponding plane of floatation.

58. *If from the centre of gravity, G, of a solid body normals are drawn to its surface of buoyancy, the points on this surface at which these lines are the normals, are the centres of buoyancy corresponding to the positions of equilibrium in which the solid can float in the liquid*

Let H be one of the points on the surface of buoyancy the normal at which passes through G GH, being perpendicular to the tangent plane at H to the surface of buoyancy, is perpendicular to the corresponding plane of floatation (Art 57) Hence, by Art 56, H is the centre of buoyancy corresponding to a position of equilibrium of the body.

Thus the problem of the determination of the positions of equilibrium of a floating body reduces to that of finding the normals to the surface of buoyancy from the point G

In the case of plane areas, normals to the curve of buoyancy are to be determined

59. We shall now determine the forms of the curve or the surface of buoyancy in some simple cases and apply (to them) the principle stated above

Ex 1. A triangular lamina floating with its plane vertical with an angular point immersed and the opposite side above the surface of the liquid.

Let ABC be a triangle (Fig 40) and let ED be the line of floatation and H the corresponding centre of buoyancy. Take AC, AB as axes of x, y, and let the coordinates of H be (x_1, y_1) Then, since H divides AL in the ratio 2 . 1, $x_1 = \frac{1}{3}AD$, $y_1 = \frac{1}{3}AE$ Now the area of the $\triangle ADE$ is constant, since it denotes the amount of the liquid displaced Let σ, ϱ denote the densities of the lamina and the liquid respectively.

$$\therefore \triangle ADE \,.\, \varrho = \triangle ABC \;\; \sigma, \text{ or } AD \;\; AE \,.\, \varrho = AB \;\; AC \;\; \sigma$$

$\therefore$ substituting,

$$x_1 y_1 = \tfrac{1}{9}AD \,.\, AE = \tfrac{1}{9}AB \;\; AC \;\; \frac{\sigma}{\varrho} = \frac{\sigma bc}{9\varrho} = k^2 \text{ (say)}$$

$\therefore$ the locus of H, or the curve of buoyancy, is the hyperbola $xy = k^2$, whose asymptotes are AC and AB ...(i)

Again, the locus of L is the hyperbola

$$xy = \tfrac{1}{4}AD \;\; AE = \tfrac{9}{4}k^2,$$

since the coordinates of L are $\frac{1}{2}AD$ and $\frac{1}{2}AE$; it has the same asymptotes AC and AB And as the line ED, between these asymptotes, is bisected at L, ED is the tangent to the hyperbola at L Therefore, the envelope of ED or the curve of floatation is the hyperbola

$$xy = \tfrac{9}{4}k^2 \qquad \ldots\ldots\ldots\ldots \text{(ii)}$$

The equation of the curve of buoyancy referred to internal and external bisectors of the angle $BAC (= \theta)$ as axis of x and y, can be found to be

$$x^2 \sec^2\frac{\theta}{2} - y^2 \operatorname{cosec}^2\frac{\theta}{2} = 4k^2 \qquad \ldots\ldots\ldots\ldots \text{(iii)}$$

The normal to this at a point (x', y') is

$$xy' \cos^2\frac{\theta}{2} + yx' \sin^2\frac{\theta}{2} = x'y'$$

Transferring the equation back to the original axes, we get the equation to the normal to the curve $xy=k^2$ at the point H or (x_1, y_1) as

$$x(x_1-y_1\cos\theta)+y(x_1\cos\theta-y_1)=x_1^2-y_1^2.$$

If this passes through G or $\left(\frac{b}{3}, \frac{c}{3}\right)$ we must have

$$b(x_1-y_1\cos\theta)+c(x_1\cos\theta-y_1)=3(x_1^2-y_1^2)$$

Let $AD=\xi$, then $x_1=\frac{1}{3}\xi$, and $x_1y_1=\frac{1}{9}AD\ AE=k^2$ Substituting in the above and simplifying,

$$\xi^4-(b+c\cos\theta)\xi^3+9k^2(c+b\cos\theta)\xi-81k^4=0. \quad \text{...(iv)}$$

The roots of this equation give the values of ξ (or the immersed portion of the side AC), and hence the positions of equilibrium Since the last term is negative it is clear that of the four roots, one or three should be negative and hence inadmissible Thus the remaining three (or one) give three (or one) positions of equilibrium

Considering the case of the isosceles triangle (cf Ex 3, Art 53), we get by putting $b=c$ in (iv),

$$\xi^4-b(1+\cos\theta)\xi^3+9k^2b(1+\cos\theta)\xi-81k^4=0,$$

or
$$(\xi^4-81k^4)-b(1+\cos\theta)\xi(\xi^2-9k^2)=0$$

$$\therefore \text{ either } \xi^2=9k^2 \text{ or } \xi^2+9k^2-b(1+\cos\theta)\xi=0. \quad \text{......(v)}$$

The first gives $\xi=3k$, whence $AE=AD=3k$ This is the symmetrical position

If the second gives possible values, then there will be the corresponding positions of equilibrium as well Obviously both the roots, if real, are positive, and we have for real roots,

$$b^2(1+\cos\theta)^2>36k^2,$$

or
$$b^2\cos^4\frac{\theta}{2}>9k^2, \ i\ e. > \frac{\sigma}{\varrho}b^2,$$

$$\therefore \cos^4\frac{\theta}{2}>\frac{\sigma}{\varrho}.$$

If this condition is satisfied there will be two more positions of equilibrium If ξ_1, ξ_2 be the two roots,

$$\xi_1\xi_2=9k^2 \text{ and } \xi_1+\xi_2=2b\cos^2\frac{\theta}{2}.$$

Since $\xi \,.\, AE=9k^2$, ξ_2 must be AE. Hence the second result can be written as

$$AD+AE=2b\cos^2\frac{\theta}{2}$$

These are in accordance with the results previously obtained wherein ϱ had been taken as unity.

Ex 2. A lamina bounded by an hyperbolic arc and a chord floats in a vertical plane with the chord above the surface of the liquid. Find the forms of the curves of buoyancy and of floatation

Let ABB' be the hyperbolic lamina, QQ' be a line of floatation and H the corresponding centre of buoyancy.

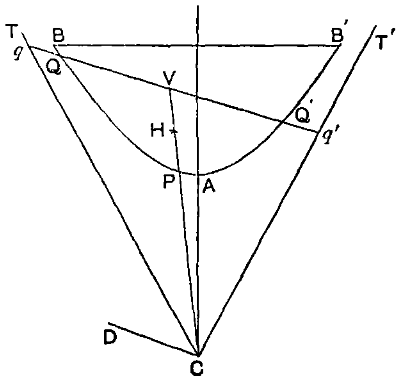


FIG. 43.

Then H lies on the diameter CPV, which bisects the system of thin strips parallel to QQ' (which are horizontal). Also the area QAQ' will be constant for all possible lines of floata-

tion, since area $QAQ' \cdot$ area $BAB' = \sigma \,.\, \varrho$ where σ and ϱ are the densities of the lamina and the liquid respectively.

To find the magnitude of the area and the position of H, we draw the diameter CD conjugate to CP, let their lengths be b' and a', and the $\angle PCD = \theta$

Taking these as axes of y and x, the equation of the hyperbola is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$$

Dividing the area AQQ' into thin strips parallel to QQ' we obtain this area

$$= \int_{a'}^{x'} 2y\,dx \sin\theta = 2\sin\theta \int_{a'}^{x'} \sqrt{\frac{x^2}{a'^2} - 1}\; b'dx$$

$$= a'\,b' \sin\theta \left[\frac{x'}{a'}\sqrt{\frac{x'^2}{a'^2} - 1} - \log\left(\frac{x'}{a'} + \sqrt{\frac{x'^2}{a'^2} - 1}\right)\right]$$

where $CV = x'$.

Now, $a'b' \sin\theta$ denotes the area of the parallelogram formed by the conjugate semi-diameters, therefore this is constant Thus the area $AQQ' =$ a constant $\times$ a function of $\frac{x'}{a'}$; and since the area is constant for all values of x' and a', it follows that x'/a' or CV/CP must be constant

Again, the distance CH is, by taking moments about CD, obtained as

$$\left[\int_{a'}^{x'} 2xy\,dx \sin\theta\right] \div \text{area } AQQ',$$

or $$2\sin\theta \int_{a'}^{x'} x\sqrt{\frac{x^2}{a'^2} - 1}\; b'dx \div \text{area } AQQ',$$

or $$\tfrac{2}{3}a'b' \sin\theta \left(\frac{x'^2}{a'^2} - 1\right)^{\frac{3}{2}} . \, a' \div \text{area } AQQ'.$$

As before, this last expression can be written as

$$a' \; f\left(\frac{x'}{a'}\right)$$

$$\therefore \; CH : a' = CH : CP = f\left(\frac{x'}{a'}\right) = \text{constant} = \lambda \text{ (say)},$$

since $x' : a'$ has been proved to be constant

To find the locus of H, let us refer the hyperbola ABB' to its asymptotes CT, CT' Its equation is then $xy = c^2$ Let the coordinates of P be (x_1, y_1) and of H (x_2, y_2). Then $\dfrac{x_2}{x_1} = \dfrac{y_2}{y_1} = \dfrac{CH}{CP} = \lambda$ Therefore

$$x_2 y_2 = \lambda^2 x_1 y_1 = \lambda^2 c^2$$

So the locus of H is another hyperbola, $xy = \lambda^2 c^2$, having the same asymptotes CT, CT'

Again, as $CV : CP$ is constant we can similarly prove that the locus of V is also an hyperbola with the same asymptotes And since qVq' is bisected at V, this line must be the tangent to this hyperbola at V. This is, therefore, the envelope of QQ', or the curve of floatation

EXAMPLES. 8.

1. A circular disc is floating in liquid with its plane vertical Show that the curves of floatation and of buoyancy are concentric circles

2. A solid hemisphere floats in water with its plane base wholly above the surface, determine the forms of the surfaces of floatation and of buoyancy.

3 A rectangular lamina $ABCD$ floats in a vertical plane with BC and parts of AB, CD immersed in the liquid. Prove that the lines of floatation pass through a definite point, and that the curve of buoyancy is a parabola whose axis passes through the middle points of AD and BC.

4. A uniform thin lamina whose shape is a parabola bounded by a double ordinate, floats in a liquid with its plane vertical and the bounding chord wholly outside the liquid. Prove that the curves of floatation and of buoyancy are equal parabolas

CHAPTER VII

STABILITY OF EQUILIBRIUM OF FLOATING BODIES

60. Suppose a solid body to be floating, partially * immersed in a liquid, in a position of equilibrium, let it be given any slight displacement and then left to move under the influence of the forces acting on it If the forces tend to bring the body back to the original position the equilibrium is said to be *stable*, if they cause the body to recede away from the original position the equilibrium is said to be *unstable* ; and if the body remains at rest in the displaced position, the equilibrium is *neutral*

To be more strict, we ought to define the stable, unstable or neutral equilibrium for the particular displacement (or type of displacement) under consideration, because the body may be stable for one displacement, but unstable for another. Such cases are often met with, in this chapter we shall discuss certain types of displacement only (see Art 62)

61. Firstly, we shall show that floating bodies are stable for vertical displacements, that is, when the body is lightly pressed down or lifted up without any angular motion In the first case the solid would displace more liquid in the new position ; therefore the fluid thrust on it will be greater in magnitude than that in the position of equilibrium, *i e* greater than the weight of the body Hence the resultant force on the body will be upwards, tending to move the body up, *i e* towards the original position

* It may be totally immersed if the fluid be heterogeneous, the definition holds in this case too

Similarly, if the body be lifted up, it would displace less liquid, and its weight being now greater than the fluid thrust would urge the body downwards, *i e.* towards the original position

The bodies are here considered to be rigid, *i e* to be incapable of changing its volume under pressures. The above would not be true, in general, for those bodies whose volumes change with their depths.

62. A general displacement consists of a rectilinear displacement together with an angular one. In the case of a floating solid we can give it a vertical and a horizontal displacement, as also an angular one about some convenient line When these displacements are small, as in the case of the present problem of studying the stability of the solid, they can be dealt with separately. We have already seen that for vertical displacements the equilibrium is stable For horizontal ones, the forces acting on the body do not undergo any change as regards their magnitudes or relative lines of action So the body remains in equilibrium in the new position as well. Thus the equilibrium is neutral for horizontal displacements

As regards angular displacements we shall confine our attention to those in which the immersed portions of the solid remain of the same volume so that there is no tendency of the fluid thrust in the displaced position to raise or lower the body. The rest of the chapter deals with such displacements We shall first show how a displacement of this kind can be given to the floating body.

63. *If the plane of floatation be turned through a small angle about a line through the centre of gravity of its area, the new position of the plane cuts off from the body the same volume as before and is therefore another plane of floatation of the body*

In other words, if the floating body be given a small angular displacement by turning it about a line in the plane

of floatation through the c g. of the area, the body will displace the same volume of liquid as before.

Let AB be the plane of floatation, C the c g of its area and ADB be the volume immersed Let $A'B'$ be another plane through Cx, a line through C in the plane AB. To prove that the vol ADB = the vol. $A'DB'$.

Take Cx and a perpendicular line CB in the plane AB as axes of x and y, so that CA is the negative side of the y-axis and y of any point P' on this side of Cx (to be denoted

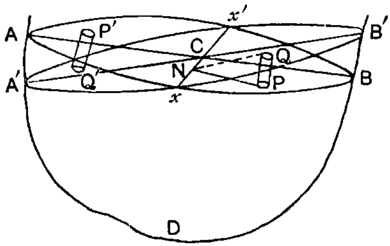


FIG 44.

by y') is negative Divide the area AB into small elements such as a at P on the positive side of Cx and a' at P' on the negative side On them draw cylinders between the planes AB and $A'B'$ Then

the vol of the wedge BCB'

$$=\text{sum of such cylinders to the right of } xx'$$

$$=\Sigma a \; PQ=\theta\Sigma ay, \quad \ldots\ldots(1)$$

where θ denotes the angle between the two planes, θ being small, $PQ=PN\tan\theta=PN\;\theta=y\theta$. Similarly,

the vol of the wedge ACA'

$$=\text{sum of the cylinders to the left of } xx'$$

$$=\Sigma a' \; P'Q'=-\theta\Sigma a'y', \quad \ldots\ldots(2)$$

since the y''s are negative on this side and the volume should be positive

Now, since the c g of the area AB is at the origin of coordinates $\Sigma ay = 0$, the summation extending over the whole area AB, this may be written as

$$\Sigma ay + \Sigma a'y' = 0,$$

where the first summation is taken over the portion xBx' and the second over xAx'.

$$\therefore \ \theta \Sigma ay = -\theta \Sigma a'y'$$

From (1) and (2) this is equivalent to

the vol of the wedge BCB' = the vol. of the wedge ACA'

Adding the vol $A'CBD$ to both sides, we get

$$\text{vol } A'DB' = \text{vol } ADB.$$

64. Metacentre Let G denote the c.g of the floating solid and H its centre of buoyancy in the position of equilibrium, then GH is vertical in that position Let

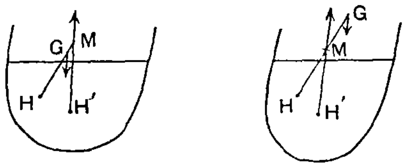


FIG 45

the body be slightly displaced so that H' is the new centre of buoyancy and let the vertical (in the new position) through H' meet GH in M Then M is called the *metacentre* of the body

The forces on the body (in the displaced position) are its weight at G and the fluid thrust at H', these constitute a couple whose moment $= g\varrho V \ GM \sin\theta$ or $g\varrho V\theta \ GM$, where V is the volume of the liquid displaced, ϱ is its density and θ is the *small* angle through which the body is

turned, viz the angle HMH' This couple is generally referred to as the *restoring couple.*

(1) *If the metacentre M is above G,* as in the left-hand diagram, the restoring couple tends to bring GH to the vertical, *i e* tends to bring the body back to its original position , hence *the equilibrium is stable*

(2) *If the metacentre is below G,* as in the right-hand figure, the restoring couple tends to increase the angle θ, *i e* causes the body to recede further from its original position , hence *the equilibrium is unstable*

(3) *If M coincides with G,* GH' is now vertical and so the condition of equilibrium is satisfied (Art. 51). The body remains at rest in the displaced position , the *equilibrium is thus neutral*

In other words, *the equilibrium is stable, unstable or neutral according as $HM >, <,$ or $= HG$.*

The length GM is sometimes called the metacentric height

65. Since $H'M$ is vertical, it is perpendicular to the plane of floatation and therefore to the tangent plane at H' to the surface of buoyancy (Art 57) Therefore $H'M$ is normal to the curve of buoyancy at H' For a similar reason HG (or HM) is the normal to the same curve at H Since H and H' are neighbouring points on the curve, we see that M is a centre of curvature of this curve Thus the theorem :

The metacentre is the centre of curvature to the curve of buoyancy at the point which corresponds to the position of equilibrium

66. Let us consider, as an example, the case of an isosceles triangular lamina floating in a vertical plane with its vertex immersed and the base out of the liquid (cf Ex 1, Art 59, and Ex 3, Art 53)

The curve of buoyancy was shown to be

$$xy = k^2 = \frac{\sigma}{9\varrho} b^2,$$

or
$$x^2 \sec^2 \frac{\theta}{2} - y^2 \operatorname{cosec}^2 \frac{\theta}{2} = 4k^2,$$

according as its asymptotes or its principal axes were chosen to be the axes of coordinates, so that the semi-axes of the hyperbola are

$$a' = 2k \cos \frac{\theta}{2}, \quad b' = 2k \sin \frac{\theta}{2}$$

Now the radius of curvature at the point H or (x_1, y_1) according to the first system of axes is given by

$$\frac{(b'^2 - a'^2 + r^2)^{\frac{3}{2}}}{a'b'},$$

where r is the distance of the point from the centre So

$$r^2 = x_1{}^2 + y_1{}^2 + 2x_1 y_1 \cos \theta = \frac{4b^2}{9} \cos^4 \frac{\theta}{2} - 4k^2 \sin^2 \frac{\theta}{2}$$

Substituting, we get the radius of curvature for the equilibrium-position, or,

$$HM = \frac{2b^3}{27k^2} \cot^2 \frac{\theta}{2} \sin \frac{\theta}{2} \left(\cos^2 \frac{\theta}{2} - \frac{9k^2}{b^2} \right)^{\frac{3}{2}},$$

using the conditions $x_1 + y_1 = \frac{2}{3} b \cos^2 \frac{\theta}{2}$ and $x_1 y_1 = k^2$ (Ex 1, Art. 59)

Also,

$$HG^2 = \left(\frac{b}{3} - x_1 \right)^2 + \left(\frac{b}{3} - y_1 \right)^2 + 2 \left(\frac{b}{3} - x_1 \right) \left(\frac{b}{3} - y_1 \right) \cos \theta,$$

whence
$$HG = \frac{2b}{3} \sin \frac{\theta}{2} \left(\cos^2 \frac{\theta}{2} - \frac{9k^2}{b^2} \right)^{\frac{1}{2}}$$

The equilibrium will be stable, neutral or unstable according as

$$HM >, =, \text{ or } < HG,$$

or as $$\cos^2\frac{\theta}{2}\left(\cos^2\frac{\theta}{2}-\frac{9k^2}{b^2}\right)>,\ =,\ \text{or} <\frac{9k^2}{b^2}\sin^2\frac{\theta}{2},$$

or as $$\cos^4\frac{\theta}{2}>,\ =,\ \text{or} <\frac{9k^2}{b^2} \text{ or } \frac{\sigma}{\varrho}$$

But it has been shown in the example referred to that for equilibrium $\cos^4\frac{\theta}{2}>\frac{\sigma}{\varrho}$. Hence the unsymmetrical position of equilibrium is *stable*

For the symmetrical position of equilibrium, we have $x_1=y_1=k$

$\therefore$ HM (or the radius of curvature)

$$=2k\frac{\sin^2\frac{\theta}{2}}{\cos\frac{\theta}{2}}=2k\left(\sec\frac{\theta}{2}-\cos\frac{\theta}{2}\right).$$

And $$HG=\tfrac{2}{3}b\cos\frac{\theta}{2}\left(1-\frac{3k}{b}\right)=\tfrac{2}{3}b\cos\frac{\theta}{2}-2k\cos\frac{\theta}{2}$$

Since $\cos^2\frac{\theta}{2}>\frac{3k}{b}$ for equilibrium, $\frac{2b}{3}\cos\frac{\theta}{2}>2k\sec\frac{\theta}{2}$

$$\therefore\ HM<HG.$$

Thus the equilibrium is *unstable*

If, however, the unsymmetrical positions of equilibrium are not possible, we must have

$$\cos^4\frac{\theta}{2}<\frac{\sigma}{\varrho} \text{ or } \cos^2\frac{\theta}{2}<\frac{3k}{b}$$

$\therefore$ for the symmetrical position, $HM>HG$ or the equilibrium is *stable*

67. The method indicated in the previous article generally entails elaborate calculations, and is therefore seldom resorted to The elegant formula which we are now going to establish provides an easy mode of determining the value of HM in many cases. The length HG is cal-

culated, as before, from the knowledge of the coordinates of H and G.

Let G and H denote the centres of mass of the whole solid and of the volume V immersed in the liquid in the position of equilibrium ; let a vertical plane through G, H be a plane of symmetry of the body (or at least of the part of the body immersed in the liquid and of the adjoining portions) If the body be given a small angular displacement about the line Cx through the c g , C, of the plane of floatation (Art. 63), *which is perpendicular to the vertical plane of symmetry, the length HM (for this displacement) $= Ak^2/V$, where Ak^2 is the moment of inertia of the plane of floatation about Cx*

The condition that the vertical plane through G, H is a plane of symmetry ensures that in the displaced position the new centre of buoyancy, viz. H', should be in this plane as also the c g , C, of the plane of floatation

Let the plane of the paper be the plane of symmetry and let Cx be the line, perpendicular to this plane in the plane of floatation, ACB, corresponding to the position of equilibrium. Let $A'CB'$ be the plane of floatation in the new position and θ the angle between the planes (which intersect along Cx) The line Cx is not indicated in the diagram. Also, let ρ be the density of the liquid

Draw $H'M$ perpendicular to the plane $A'CB'$ to meet HG in M , then M is the metacentre* and the angle $HMH' = \theta$

Let g and f denote the c g of the equal wedges ACA' and BCB' (Art. 63). The forces acting on the body in the displaced position constitute a couple of moment $g\rho V\theta \,.\, GM$ (Art 64) about an axis perpendicular to the plane of the paper. Since moments of a couple about parallel axes are equal, the moment of this couple about the parallel axis through G must have the same value.

* In order that M may exist it is necessary that the vertical line through H' should intersect HG ; this is assured if the vertical plane through G, H is a plane of symmetry and so contains H' for *any* small value of θ.

We shall next find another expression for this moment which is equal to the sum of the moments of the weight of the body through G and the fluid thrust through H'. The former force has no moment about the axis through G.

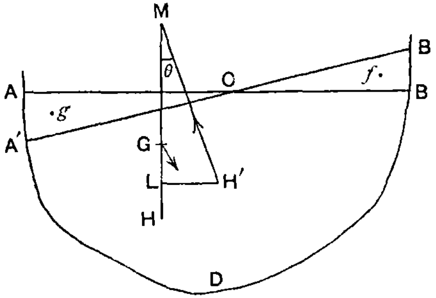


FIG 46

Now, the fluid thrust (reversed in direction) is equal to the wt of the vol $A'DB'$, which is equal to the wt. of the vol ADB + the wt of the wedge BCB' − the wt of the wedge ACA'. Let v denote the vol of each wedge.

∴ from statical principles, the fluid thrust (which acts upwards) is the resultant of a force $g\varrho V$ (= wt. of ADB) acting upwards through H, a force $g\varrho v$ (= wt of BCB') acting upwards through f and a force $g\varrho v$ (= wt. of ACA') acting downwards through g.

∴ moment of the fluid thrust = algebraical sum of the moments of the three latter forces.

The moment of the first of these forces

$$= -g\varrho V \ HG \sin\theta = -g\varrho V\theta \, . \, HG, \text{............(1)}$$

since it is in opposite direction to that of the fluid thrust

The other two forces form a couple whose moment is equal to its moment about Cx, a parallel axis To determine the latter we take in the plane ACB the perpendicular lines Cx and ACB as axes of x and y respectively, CB denoting

the positive direction of y We next divide, as in Art 63, the area AB (see Fig 44) into elements like a at P on the positive side of y, and a' at P' on the negative side. Then the vol. of the wedge $BCB' = \Sigma ay\theta$ and that of $ACA' = -\Sigma a'y'\theta$. [See (1), (2) of Art 63]

$\therefore$ the moment of the upward force $g\varrho v$ through f
$=$ sum of moments of upward forces $\Sigma g\varrho\theta ay$, at respective elements,
$= g\varrho\theta\Sigma ay^2$,

and the moment of the downward force $g\varrho v$, through g,
$=$ sum of moments of downward forces, $-\Sigma g\varrho\theta a'y'$, at respective elements,
$= g\varrho\theta\Sigma a'y'^2$,

since it is in the same sense and y' is negative.

$\therefore$ their sum $= g\varrho\theta\{\Sigma ay^2 + \Sigma a'y'^2\}$

$$= g\varrho\theta \; Ak^2, \qquad \text{......(2)}$$

where Ak^2 is the moment of inertia of the area AB about Cx, by definition Therefore adding (1) and (2) we get the restoring couple to be

$$g\varrho\theta(Ak^2 - V.HG) \qquad \text{......(3)}$$

Equating it to the former value, viz $g\varrho\theta V\,GM$, and removing the common factor $g\varrho\theta$, we have

$$V\,GM = Ak^2 - V\,HG$$

$$\therefore HG + GM \text{ or } HM = \frac{Ak^2}{V} \qquad \text{......(4)}$$

68. The following is another proof of the above theorem due to Besant and Ramsay [*Hydromechanics*, Pt. I, 7th edition, pp 103-4]

The plane of symmetry, in this proof also, is taken as the plane containing the points G, H, H' and M which have the same significance as in the last article. The

points G and M are not shown in the accompanying diagram for the sake of a clear figure ; G lies on HL produced and M is the intersection of HL and the line through

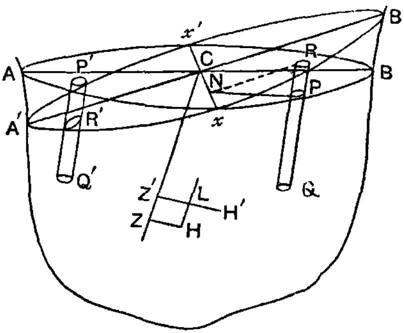


FIG 47.

H' perpendicular to the plane $A'B'$ The lines Cx, CB and CZ (perpendicular to the plane AB) are taken as axes of x, y, z respectively, CZ being taken downwards and CB the positive side of y. Draw HZ, $H'Z'$ perpendiculars to CZ, and HL parallel to CZ meeting $H'Z'$ in L

Divide the area AB into elements such as a at P on the positive side of y and a' at P' on the negative side On these elements draw cylinders having their axes parallel to CZ, i e perpendicular to AB, and let them meet the surface of the immersed solid in Q, Q', and the plane of $A'B'$ in R, R' respectively. Then the volume cut off by AB

$$= \text{sum of cylinders like } PQ \text{ and } P'Q'$$
$$= \Sigma a \,.\, PQ + \Sigma a' \,.\, P'Q',$$

whilst the volume cut off by $A'B'$

$$= \text{sum of cylinders like } RQ \text{ and } R'Q'$$
$$= \Sigma a \,.\, RQ + \Sigma a' \,.\, R'Q',$$

the first term of each sum representing the summation over the area xBx' and the second term over the area xAx'

$\therefore$ moment of the first volume about Cx

$$=\Sigma ay \cdot PQ+\Sigma a'y'.P'Q',$$

y, y' being the ordinates of P, P', the second term is clearly negative as it should, since y''s are negative

$$\therefore\ V \cdot HZ=\Sigma ay \cdot PQ+\Sigma a'y' \cdot P'Q', \ldots\ldots\ldots(1)$$

since H is the c g of this volume * Similarly we shall have from the second volume,

$$V.H'Z'=\Sigma ay \cdot RQ+\Sigma a'y'.R'Q' \quad \ldots\ldots\ldots(2)$$

$\therefore$ by subtraction,

$$V \cdot (H'Z'-HZ)=\Sigma ay \cdot PR-\Sigma a'y'.P'R' \quad \ldots\ldots(3)$$

But $PR=PN \cdot \tan\theta=y \cdot \theta$ and $P'R'=-y'\theta$, since y' is negative, but $P'R'$ is taken as positive in the equation (3)

$$\therefore\ V \cdot H'L=\theta\Sigma ay^2+\theta \cdot \Sigma a'y'^2$$
$$=\theta[\Sigma ay^2+\Sigma a'y'^2]$$
$$=\theta \cdot Ak^2, \qquad (4)$$

from definition And $H'L=H'M\sin\theta=H'M.\theta=HM.\theta$, since θ is small and H, H' are quite close to one another

Therefore $$HM=\frac{Ak^2}{V} \quad \ldots\ldots\ldots\ldots (5)$$

69. The position of the metacentre, or what comes to the same thing, the metacentric height (viz GM), may be determined experimentally as follows

A part of the floating body (or ship) of known weight w, which is purposely designed to be detachable, is shifted from the original position to another through a horizontal

* The weights of the elements may be supposed to be all acting parallel to CZ, as this does not affect the position of the c g of the volume If the portion immersed has some bulging part, this would contribute equally to the right-hand sides of (1) and (2), and therefore would not influence the equation (3)

distance x and a vertical distance y. To be definite, let the new position be lower than the former. These distances are measured so that x and y are known quantities In consequence of this shifting, the body will turn through a small angle, although it continues to displace the same amount of the liquid. The measure of this angle can be found by noting the shifting of a plumb line (*i.e* a string supporting a heavy object) relative to the floating body Let it be θ ; if w be small compared with the total weight W of the body, θ will be small

Let A and B be the initial and final positions of the weight w, and AZ, AZ' the initial and final directions of the plumb line To see the effect of the transference of the weight w from A to B, let us introduce (in the final position) two equal and opposite forces, each $= w$, along AZ'. Now the downward force, w, at A along with the weight of the remainder of the floating body will give the total weight W acting through the original c g. of the whole body The upward force at A together with the weight w at B forms a couple whose moment is easily seen to be

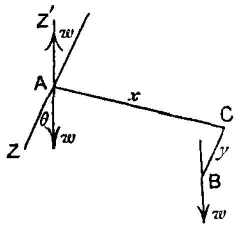


FIG 48

$$w(x \cos \theta - y \sin \theta) \text{ or } w(x - y\theta)$$

Thus the shifting of the weight w is equivalent to the introduction of the couple $w(x - y\theta)$, acting on the whole body This will tend to displace the body from the first position of equilibrium , and in the new position of equilibrium it will be balanced by the restoring couple, $W . GM . \theta$ (Art. 64).

$$\therefore W \; GM \; \theta = w(x - y\theta),$$

whence GM can be calculated.

70. Applications. *Ex.* 1. A solid cone, of semi-vertical angle α, height h and sp. gr σ, floats in equilibrium in a liquid of sp gr ϱ with its axis vertical Determine whether the equilibrium is stable or unstable

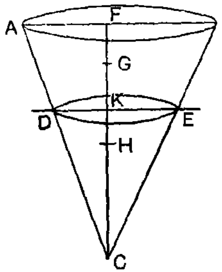


FIG 49

Let $h' = CK$ be the length of the axis immersed. For equilibrium, the weight of the vol CDE of liquid = the weight of the cone ; or

$$\tfrac{1}{3}\pi h'^3 \varrho \tan^2\alpha = \tfrac{1}{3}\pi h^3 \sigma \tan^2\alpha,$$

or
$$h'^3 \varrho = h^3 \sigma \quad \ldots\ldots\ldots\ldots(1)$$

Here the plane of floatation is a circle whose c g is the centre The displacement can, therefore, be given about any diameter

$\therefore$ Ak^2 = moment of inertia of this circular area about the diameter

$$= \frac{\pi h'^4 \tan^4\alpha}{4}, \text{ and } V = \tfrac{1}{3}\pi h'^3 \tan^2\alpha$$

$$\therefore\ HM = \frac{Ak^2}{V} = \tfrac{3}{4}h' \tan^2\alpha$$

Obviously, $$HG = \tfrac{3}{4}h - \tfrac{3}{4}h'.$$

$\therefore$ the equilibrium will be stable, neutral or unstable according as

$$HM >, \ =, \text{ or } < HG,$$

or as $$h' \tan^2\alpha >, \ =, \text{ or } < h - h',$$

or as $$h' \sec^2\alpha >, \ =, \text{ or } < h,$$

or as $$h'^3 >, \ =, \text{ or } < h^3 \cos^6\alpha,$$

or as $$\frac{\sigma}{\varrho} >, \ =, \text{ or } < \cos^6\alpha$$

Ex 2 An uniformly thin isosceles triangular lamina (sp gr σ) is floating in water with its plane vertical and vertex immersed, the base remaining above the surface To consider the stability or otherwise of its equilibrium

We have already seen that besides the symmetrical position of equilibrium there can be two other (unsymmetrical) positions of equilibrium if $\sigma < \cos^4\alpha$, where 2α is the vertical angle of the lamina (Ex 3, Art. 53). Let the elements of the triangle be the same as in that example (see Fig 40)

(i) Consider at first the unsymmetrical position. Taking AC, AB as axes of x, y we get the coordinates of G to be $(\frac{1}{3}a, \frac{1}{3}a)$ and those of H, $(\frac{1}{3}AD, \frac{1}{3}AE)$.

$$\therefore\ HG^2 = \tfrac{1}{9}(a - AD)^2 + \tfrac{1}{9}(a - AE)^2 + \tfrac{2}{9}(a - AD)(a - AE)\cos 2\alpha.$$

Using the results, $AD + AE = 2a\cos^2\alpha$ and $AD \,.\, AE = \sigma a^2$ of the example cited, and simplifying we get (cf. Art 66),

$$HG = \tfrac{2}{3}a \sin\alpha\,(\cos^2\alpha - \sigma)^{\frac{1}{2}}. \qquad \text{(i)}$$

Here the plane of floatation is the line DE ,* it is therefore to be displaced about its middle point L, which is its c g. Moment of inertia of this line about an axis bisecting it at right angles

$$= ED \,.\, \frac{EL^2}{3} = \tfrac{1}{12}ED^3$$

$$= \tfrac{1}{12}(AE^2 + AD^2 - 2AE \,.\, AD\cos 2\alpha)^{\frac{3}{2}}$$

$$= \tfrac{2}{3}a^3\cos^3\alpha\,(\cos^2\alpha - \sigma)^{\frac{3}{2}}.$$

$$V = \text{area } AED \text{ (in this case)} = \sigma a^2 \cos\alpha \sin\alpha.$$

$$\therefore\ HM = \frac{Ak^2}{V} = \frac{2}{3}\,\frac{a\cos^2\alpha}{\sigma\sin\alpha}(\cos^2\alpha - \sigma)^{\frac{3}{2}}. \qquad \text{(ii)}$$

Now $\quad HM > HG$, if $\dfrac{\cos^2\alpha}{\sigma\sin\alpha}(\cos^2\alpha - \sigma) > \sin\alpha$,

or if $\qquad \cos^2\alpha(\cos^2\alpha - \sigma) > \sigma\sin^2\alpha$,

or if $\qquad \cos^4\alpha > \sigma$,

which is true. Hence the equilibrium is stable.

* To be accurate it is a very narrow rectangle whose breadth equals the thickness of the lamina On this hypothesis also the result will be the same as shown above.

(ii) Let us consider the symmetrical position. If E_1D_1 denote the line of floatation in this case, $AE_1 = AD_1$ and

$$\tfrac{1}{2}AE_1 \cdot AD_1 \sin 2\alpha = \tfrac{1}{2}a^2 \sin 2\alpha \,.\, \sigma,$$

whence $$AE_1{}^2 = a^2\sigma.$$

HG is clearly equal to $\frac{2}{3}(a - AE_1)\cos\alpha$, or to

$$\tfrac{2}{3}a(1 - \sqrt{\sigma})\cos\alpha.$$

And $$HM = \frac{Ak^2}{V} = \frac{\frac{1}{12}E_1D_1{}^3}{\frac{1}{2}AE_1 \cdot AD_1 \sin 2\alpha} = \tfrac{2}{3}a\sqrt{\sigma}\frac{\sin^2\alpha}{\cos\alpha}$$

$\therefore$ equilibrium is stable, neutral or unstable according as

$$\sqrt{\sigma}\sin^2\alpha >,\ =,\ < (1 - \sqrt{\sigma})\cos^2\alpha,$$

or as $$\sqrt{\sigma} >,\ =,\ < \cos^2\alpha,$$

or as $$\sigma >,\ =,\ < \cos^4\alpha$$

Thus we see that when there are three positions of equilibrium, i.e. when $\sigma < \cos^4\alpha$, the symmetrical position is unstable. When the other positions are not possible, it is stable or neutral as the case may be.

EXAMPLES. 9.

Determine the values of HM and discuss the stability or otherwise of the following solids floating in equilibrium.

1. A right circular cylinder (sp. gr. σ) floating in a liquid (sp. gr. ρ) with axis vertical; obtain the condition in terms of σ, ρ and the lengths of the axis and the radius of the cylinder.

2. A right circular cylinder, floating in water with its axis on the surface; consider displacement about a line perpendicular to the axis.

3. A right cone (sp. gr. σ) whose base is an ellipse of semi-axes a, b and whose altitude is h, floating in water with its axis vertical and vertex downwards; consider displacements about (i) the major axis, (ii) the minor axis, and (iii) a diameter making angle θ with the major axis, of the plane of floatation.

4. A right cylinder whose base is an ellipse of semi-axes a, b, floating with its axis on the surface of water and the minor axis of the base vertical; consider displacements about (i) the axis of the cylinder, and (ii) a line perpendicular to this

5. A solid sphere of radius a floating in a liquid with its centre at a height c ($<a$) above the surface

6. Show that in the case of a right circular cylinder of radius a and height h, floating with its axis vertical in any liquid, the equilibrium will be stable whatever be the specific gravities if $a\sqrt{2} > h$.

7. A solid paraboloid (sp. gr σ) floats in a liquid (sp gr. ϱ) with its axis vertical and vertex downwards. Prove that HM is constant, and that the equilibrium will be stable if

$$3a \, . \, h > \left(1 - \sqrt{\frac{\sigma}{\varrho}}\right),$$

where $4a$ is the latus rectum of the generating parabola and h is the height of the solid.

8. Find the metacentre of a square lamina (sp. gr σ) floating in a vertical plane with (i) one edge, (ii) one diagonal vertical. Discuss the stability of equilibrium of all possible cases

9. A laden ship displaces W tons of water and its metacentric height is a Some deck cargo of weight w tons is moved across the deck and the ship tilts through a small angle θ. This cargo is next removed to a lower deck, just below its second position and at a depth h below it Find through what angle the ship now tilts

10. A thin uniform lamina whose shape is that of an isosceles triangle floats in water with its plane vertical and the base horizontal and below the surface Prove that the equilibrium is stable if the sp. gr. of the lamina is $< 1 - \cos^4 a$, where $2a$ is the vertical angle of the triangle.

11. Show that the case of a triangular prism floating in a liquid with its lateral edges horizontal can be decided (for displacements about a line parallel to the lateral edges) from that of a triangular lamina floating in a vertical plane

12. A right prism whose base is an equilateral triangle floats in water with the lateral edges horizontal and only one of them *below* the surface. Show that the equilibrium is stable for all displacements in which the lateral edges remain horizontal if it be given that the sp gr of the prism $> \frac{9}{16}$

Also prove that the equilibrium will be stable for perpendicular displacements as well if the length of this prism be greater than $\frac{3}{4}$ times the side of the base.

13 A right circular cylinder floats in a liquid with its axis (of length h) horizontal and at a height c above the surface Compare the specific gravities of the cylinder and the liquid, and show that the equilibrium will be stable if $h^2 > 4(a^2 - c^2)$, where a is the radius of the base of the cylinder

14. A frustum of a cone floats, with its axis vertical, in a liquid of twice its density. Prove that the equilibrium will be stable if $(m-1)h^2 < (a-b)^2$, where

$$m = \sqrt{2} \quad (a^4 + b^4)/(a^3 + b^3)^{\frac{4}{3}},$$

h is the height of the frustum and a, b the radii of its ends.

15. A solid in the shape of two equal cones placed with their vertices coincident and axes in the same straight line, floats in a liquid of double its density, the common axis being horizontal Prove that the equilibrium is stable if the semi-vertical angle of each cone is less than 60°

16 A solid in the shape of two equal cones placed with their bases coincident floats in a liquid with the common axis in the surface of the liquid Find the limits between which the vertical angle of each cone must lie so that the equilibrium be stable

17. In Question 32, *Examples* 7 show that the equilibrium of the buoy would be stable if $2h < (\sqrt{5} - 1)a$.

CHAPTER VIII

GENERAL THEOREMS ON FLUID PRESSURE. ROTATING LIQUIDS

71. Suppose a mass of fluid, homogeneous or heterogeneous, is at rest under the action of some given forces which act throughout the volume of the fluid. We take the force on a small particle of mass m of the fluid, to be of magnitude mF where F is a function of the position of the particle. Thus, if (x, y, z) be the coordinates of the particle, F would be a function of x, y, z The components of mF parallel to the coordinate axes shall be denoted by mX, mY, mZ respectively

Since the particle (or the element) of mass m is at rest under the influence of the external force mF and the resultant fluid thrust of the surrounding fluid on it, it is clear that the latter is equal to mF in magnitude and has the same line of action (but in opposite direction) as the force mF Therefore, *the theorems which will be proved in the succeeding articles might be interpreted as relating to the fluid thrust on the particle instead of the external force acting on it*

The fluid thrust of the surrounding fluid on the particle is sometimes spoken of as the resultant fluid thrust at the point

The difference between this fluid thrust at the point and the fluid pressure at the same point should carefully be noted. Firstly, the former acts on the particle of the fluid (at the point) in a definite direction, whereas the latter acts normally on *any* element of plane area taken through the point Secondly, the magnitudes of the two are not equal,

the former being small and the latter having some finite measure

72. Let P be the point (x, y, z) referred to a definite system of coordinate axes, take a line PQ, of very small length δx, parallel to Ox and round PQ as axis, describe a cylinder of very small cross-section α Consider, now, the element of the fluid enclosed within this cylinder; let its mass be denoted by m The forces acting on it are mX, mY, mZ parallel to the axes

Let p and ρ denote the fluid pressure and density at the point P, they will vary from point to point, and are therefore functions of the coordinates (x, y, z) of P The pressure at Q will be $p + \frac{\partial p}{\partial x}\delta x$, up to first order of small quantities, because in passing from P to Q only x has varied and $PQ = \delta x$. This can also be shown in the following manner Let $p = f(x, y, z)$, then pressure at Q or $(x + \delta x, y, z)$ will be $f(x + \delta x, y, z)$, or

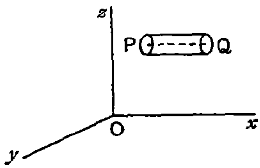


FIG 50

$$f(x, y, z) + \frac{\partial}{\partial x} f(x, y, z)\,\delta x, \text{ or } p + \frac{\partial p}{\partial x}\delta x,$$

up to first order, by Taylor's theorem.

Since α is small, fluid pressure on ends at P and Q are $p\alpha$ and $\left(p + \frac{\partial p}{\partial x}\delta x\right)\alpha$ respectively, on the curved surface the pressure is everywhere at right angles to PQ Therefore the component of the fluid pressures of the surrounding fluid on this element, in the direction of Ox,

$$= p\alpha - \left(p + \frac{\partial p}{\partial x}\delta x\right)\alpha = -\frac{\partial p}{\partial x}\alpha\,\delta x \quad \ldots \quad \ldots . (1)$$

Next, the mass of the element, viz. $m = \rho\alpha\,\delta x$, since the density may be supposed to be the same throughout this

element. As the element is in equilibrium the component of the external force, viz mX, must balance (1). Hence we have

$$\rho a\,\delta x\,.\,X - \frac{\partial p}{\partial x}\,a\,\delta x = 0,$$

or

$$\frac{\partial p}{\partial x} = \rho X \quad \text{.........................(2)}$$

Note. *To be strictly accurate*, we should take the pressure on the end at P to be $(p+\varepsilon)\,a$ and not pa (see Art 4), where ε is a small quantity of first order, the mean pressure on the end at Q will therefore be $(p+\varepsilon) + \frac{\partial}{\partial x}(p+\varepsilon)\,\delta x$, and the product of this with a will give the pressure on the end at Q

$\therefore$ we shall have, instead of (1), the component of the fluid pressures

$$= -\frac{\partial}{\partial x}(p+\varepsilon)\,a\,\delta x = -\frac{\partial p}{\partial x}\,a\,\delta x$$

up to third order, a, being an element of the area, is taken to be of second order and $\frac{\partial \varepsilon}{\partial x}$ is of first order, and therefore $\frac{\partial \varepsilon}{\partial x}\,a\,\delta x$ is of fourth order.

Again, the density ρ is not necessarily uniform throughout the element Therefore the mean density of the element would be $\rho + k$ where k is a small quantity of first order.

$$\therefore\ mX = (\rho + k)\,a\,\delta x \quad X = \rho a\,\delta x\,.\,X$$

up to third order Equating these values we arrive at the result (2).

73. Similarly we can obtain the following equations:

$$\frac{\partial p}{\partial y} = \rho Y \text{ and } \frac{\partial p}{\partial z} = \rho Z. \quad \text{.................(3)}$$

$$\therefore\ \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = \rho\,(X\,dx + Y\,dy + Z\,dz),$$

or

$$dp = \rho\,(X\,dx + Y\,dy + Z\,dz) \quad \text{.................(4)}$$

Let the system of external forces acting on the given mass of the fluid be conservative, then, by definition of conservative forces, $X\,dx + Y\,dy + Z\,dz$ must be a perfect differential, say dV where V is a function of x, y, z.

$$\therefore\ dp = \rho\, dV \quad \ldots\ldots\ldots\ldots\ldots(5)$$

74. The equation (2) is of fundamental importance in the theory of equilibrium of fluids The direction of Ox being at our disposal we can state the result as follows

At any point the rate of change of fluid pressure in any direction is equal to the product of the component of the external force (on the element) in that direction and the density of the fluid at that point

Since a force has no component in the direction at right angles to it, it follows that *in the direction at right angles to the force mF* (Art 71) *there is no change of fluid pressure*

The locus of points within the mass of the fluid under consideration, where the fluid pressure has the same value, is called a *surface of equal pressure* A section of this surface may be called a curve of equal pressure

Let P, Q be two points sufficiently near each other, on such a surface so that the straight line PQ lies on the surface Since pressures at P and Q are equal, there is no change of fluid pressure in the direction PQ. Therefore, by the theorem stated just before, PQ must be at right angles to the direction of the resultant force, mF. This important deduction can be put in the form

The external force (or the resultant fluid thrust, see Art. 71) *at any point of a mass of fluid at rest under the action of given forces, is perpendicular to the surface of equal pressure drawn through the point.*

Another proof of this theorem is given in the next article

75. Let P be a definite point on a surface AB of equal pressure, take an adjoining point Q on the same surface

so that PQ is small. Describe a cylinder of very small cross-section about PQ as axis Consider the equilibrium of the element of fluid enclosed within this cylinder. It is clear that for equilibrium the component of fluid pressures must balance the component of the external force, in the direction PQ The former is zero since the pressures on ends P and Q are equal and opposite, and that on the curved surface is normal to PQ , the latter component must, therefore, necessarily be zero, which shows that the external force acts at right angles to PQ

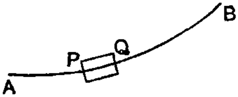


FIG. 51

Similarly, it can be shown that the external force (or the resultant fluid thrust at the point) is perpendicular to any other line PQ' on the surface AB Hence it must act normally to the surface itself

76. *Two surfaces of equal pressure cannot intersect*, since otherwise the fluid pressure will have simultaneously two different values at any point on their intersection, one corresponding to the value at each surface

Let A, B denote two surfaces of equal pressure quite close to each other Let the pressures at points on these surfaces be p and $p + \delta p$ respectively, the values of pressures obviously differing by a small quantity Take a point P on A and draw PQ normal to A meeting B in Q. Then, by the previous article, the resultant force F (per unit mass) acts along PQ Let PQ be denoted by δs From the equation (2), writing s for x, we have

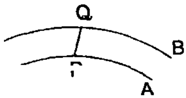


FIG 52

$$\frac{dp}{ds} = \varrho F, \text{ or } \delta p = \varrho F \, \delta s,$$

provided that δs and δp are sufficiently small, *i e* provided that the surfaces A and B are sufficiently close to each other Hence

The distance between two adjoining surfaces of equal pressure varies as the difference of pressure directly and as the density and the external force (F) *inversely*

77. The locus of points at which the density of the fluid is the same is known as a *surface of equal density.*

It has been shown in Art. 73 that if the forces acting on the fluid mass be conservative $dp = \rho\, dV$. This may be written as

$$\frac{dp}{\rho} = dV.$$

The right-hand side of this equation is a perfect differential, therefore the left-hand side must also be a perfect differential This requires that ρ should be a function of p, or p should be a function of ρ. Therefore, if on a surface p has a constant value, ρ must be constant on the same surface and *vice versa* That is, *surfaces of equal pressure are also surfaces of equal density and conversely.*

78. As an application of the results of the Articles 72 and 73, let us consider the case of a mass of homogeneous liquid at rest under gravity The force on the element of mass m is now mg, so that $F = g$ and is acting in a vertically downward direction Taking the z-axis vertically downwards, we get $X = 0$, $Y = 0$, $Z = g$. Hence (4) becomes *

$$dp = g\rho dz.$$

$\therefore$ integrating, $$p = g\rho z + C,$$

where C is some constant If z be measured from the free surface where pressure $= 0$, then at the free surface

$$0 = g\rho\ 0 + C, \text{ or } C = 0.$$

$$\therefore\ p = g\rho z$$

* This could be obtained from (2); we remark that since the external force acts wholly parallel to Oz, the variation of p in this direction is total. Therefore

$$\frac{dp}{dz} = g\rho, \text{ or } dp = g\rho\, dz,$$

If, however, the pressure at the free surface be Π, then we shall have
$$\Pi = g\rho . 0 + C, \text{ or } C = \Pi.$$
$$\therefore\ p = g\rho z + \Pi$$

These are in accordance with the results with which we are already familiar.

The case of a fluid, in general, at rest under gravity would give
$$dp = g\rho\, dz,$$

as before, but, as ρ is not a constant we cannot integrate directly. If we are given the law of density, we can thence get a differential equation in p and z, which we can integrate

Suppose there are n liquids of densities $\rho_1, \rho_2, \ldots \rho_n$ and of depths $h_1, h_2, \ldots h_n$ from the top Taking the origin on the free surface of the topmost liquid we see that the law of density is · $\rho = \rho_1$ from $z = 0$ to h_1, $\rho = \rho_2$ from $z = h_1$ to $h_1 + h_2$, etc.
$$\therefore\ p = \int_0^{h_1} g\rho_1\, dz + \int_{h_1}^{h_1+h_2} g\rho_2\, dz + \ldots + C$$
$$= g\rho_1 h_1 + g\rho_2 h_2 + \ldots + C,$$

a result which is the same as obtained in Art. 9.

The case of the atmosphere will be considered in the next chapter.

79. *Ex.* Assuming that the density ρ and pressure p in a heavy compressible fluid are connected by the equation $p - p_0 = k \log \dfrac{\rho}{\rho_0}$, where p_0 and ρ_0 denote the values at the surface, prove that the density at any depth is to the density at the surface as $k . k - p'$, where p' is what the increment of the pressure would be at this depth if the density were constant and equal to that at the surface.

Since the fluid is heavy, it is implied that it is at rest under gravity.

$\therefore$ we have
$$dp = g\rho\, dz$$

But $p - p_0 = k \log \frac{\varrho}{\varrho_0}$, $\therefore\ dp = \frac{k}{\varrho} d\varrho$

Substituting in the former, $\frac{d\varrho}{\varrho^2} = \frac{g}{k} dz$.

$$\therefore\ \frac{1}{\varrho} = -\frac{gz}{k} + C$$

At the surface, where $z = 0$, $\varrho = \varrho_0$,

$$\therefore\ \frac{1}{\varrho} = \frac{1}{\varrho_0} - \frac{gz}{k}$$

Whence we get $\qquad \varrho : \varrho_0 = k : k - gz\varrho_0$.

Now, if the density were constant and equal to ϱ_0

$$dp = g\varrho_0 \, dz,$$

$$\therefore\ p = g\varrho_0 z + p_0',$$

where p_0' denotes the pressure in that case at the surface

$\therefore\ g\varrho_0 z = p - p_0' =$ increment of pressure, proving the proposition

EXAMPLES. 10.

1. If a liquid, at rest under gravity, be heterogeneous, the density at depth z being kz, show that the pressure at this depth is $P + \frac{gkz^2}{2}$, where P is the atmospheric pressure.

✓2. In a fluid at rest under gravity, the pressure varies as the nth power of the density at any point (n being > 1), and at the free surface the pressure is zero, prove that the pressure at depth z varies as $z^{\frac{n}{n-1}}$.

3. The particles of a sphere of homogeneous fluid of radius a are attracted to the centre with forces inversely proportional to the distance from the centre, prove that the pressure at a distance x from the surface (measured along the radius) is

$$C + D\varrho\left(x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \cdot\ \right),$$

where C is a constant, D is the force per unit mass at the surface of the sphere and ϱ the constant density of the fluid.

4. A straight rod AB, every particle of which attracts with a force varying inversely as the square of the distance, is surrounded by a mass of homogeneous incompressible fluid show that the surfaces of equal pressure are ellipsoids of revolution [Use the result that the resultant attraction of the rod AB at any point P bisects the angle APB]

5. A mass of homogeneous fluid is subject to a constant force (in magnitude and direction) together with an attraction towards a fixed point varying as the distance from that point ; prove that the surfaces of equal pressure are spherical

ROTATING LIQUID

80. It may seem strange that in a book which expressly deals with fluids at rest, the subject of moving liquids is brought in We shall not, however, discuss about liquids rotating in any manner whatsoever, but only a very special case wherein *a given mass of liquid is rotating about a vertical axis with constant angular velocity without relative motion (between its various elements) and is acted on by gravity alone* The whole mass of the liquid rotates, in this case, as if it were a solid mass , there is no change of the relative positions of the different elements throughout the motion We shall presently show that such a case might be reduced to a problem of statics, and thus justify, in a way, the inclusion of this problem.

81. Let Oy (upward vertical) be the axis about which the given mass of liquid is rotating with constant angular velocity ω. An element of the liquid, of mass m at distance x from Oy, will describe uniformly the horizontal circle PP' whose centre is N. We know, from dynamics, that a particle moving uniformly in a circle has an acceleration $\omega^2 . PN$ or $\omega^2 x$ towards the centre Therefore the forces mg (or the weight) and mF (or the resultant fluid thrust) which are actually acting on the particle, must together give rise to the above acceleration, $\omega^2 x$, towards N, *i e* are equivalent to a force $m\omega^2 x$ along PN Hence the force

$m\omega^2 x$ acting away from N would balance the forces mg and mF. So we have to consider the problem of equilibrium of the particle under these three forces

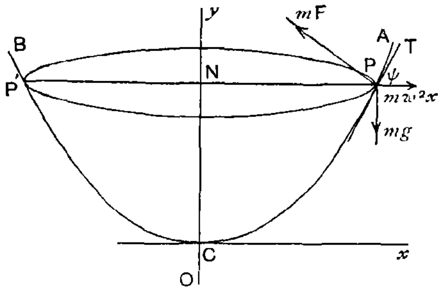


FIG 53

The surfaces of equal pressure will, from symmetry, be surfaces of revolution about Oy as axis Let the generating curve of equal pressure through P be ACB as represented in Fig 53 Then the force mF is acting along the normal to the curve ACB at P (Art 74) If ψ denotes the inclination of the tangent to the horizontal, we get, by resolving,

$$mg = mF\cos\psi \text{ and } m\omega^2 x = mF\sin\psi$$

$\therefore$ eliminating F, we get

$$\tan\psi = \frac{\omega^2 x}{g} \quad \ldots \quad (6)$$

Let the coordinates of P, regarded as a point of the curve ACB, be (x, y) referred to some suitable origin Then (6) gives

$$\frac{dy}{dx} = \frac{\omega^2 x}{g}$$

$\therefore$ integrating,

$$y = \frac{\omega^2 x^2}{2g} + C \quad \ldots\ldots (7)$$

This shows that the curve ACB is a parabola of latus rectum $\frac{2g}{\omega^2}$, which is independent of the position of P

Therefore

The surfaces of equal pressure are equal paraboloids generated by the revolution of parabolas $\left(\text{of latus rectum equal to } \frac{2g}{\omega^2}\right)$ *about the vertical axis of revolution*

If we take the origin to be on the parabola (at its vertex), C would be zero and the equation becomes

$$y = \frac{\omega^2 x^2}{2g} \qquad \qquad \qquad (8)$$

With reference to this origin, the other parabolas will be given by (7), the constant C depending on the particular curve chosen We shall denote the free surface (on which the pressure may be either zero or equal to the atmospheric pressure Π as the case may be) by (8)

82. Let OPA be the free surface whose equation is given by (8), and let the pressure at any point on it be Π. To find the pressure at any point Q within the liquid we draw the vertical line QP meeting the free surface at P, and describe a cylinder of small cross-section a about PQ Let a' be the area of the section of this cylinder by the free surface and ψ the angle between the tangent at P and Ox Then $a' \cos\psi = a$, since a is the projection of a' Let the pressure at Q be p.

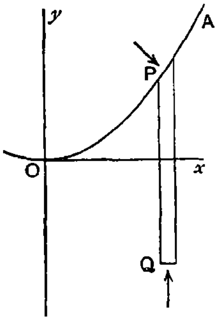


FIG. 54

Now, this cylinder of the liquid moves round Oy, but has no vertical motion. Therefore the vertical forces acting on this element will be in equilibrium They are (1) the weight, $g\varrho a . PQ$, of the cylinder, (2) pressure pa at the end Q, and

(3) the component, $\Pi a' \cos\psi$ or Πa, of the pressure at the end P.

$$\therefore\ pa = \Pi a + g\varrho \cdot PQ \cdot a,$$

or

$$p = \Pi + g\varrho \cdot PQ \qquad \text{(9)}$$

But $PQ = y_P - y_Q = \frac{\omega^2}{2g}x^2 - y$, where x denotes the common abscissa of P and Q, and y the ordinate of Q. Therefore

$$p = \Pi + g\varrho\left(\frac{\omega^2}{2g}x^2 - y\right) \qquad \text{(10)}$$

Note. The result (9) shows that the pressure at any depth below the free surface is given by the same rule as in the case of fluids at rest under gravity. The pressure on a horizontal element (of area)

$= pa = \Pi a + g\varrho \cdot PQa$

$= \Pi a +$ weight of the column of the liquid up to the free surface

$\therefore$ pressure on a horizontal area (of magnitude A)

$= \Pi A +$ weight of the column of the liquid above it up to the free surface, ...(11)

a result similar to the one established in Art 18

83. Let the equation of the curve of equal pressure through Q be

$$y = \frac{\omega^2}{2g}x^2 + C;$$

with reference to the vertex, O, of the free surface as the origin.

Substituting in (10), we get

$$p = \Pi - g\varrho C, \text{ or } C = (\Pi - p)/g\varrho \qquad \text{(12)}$$

This gives the meaning of the constant C in the equation (7)

84. We shall next show how the equation (4) of Art. 73 can give the same results as have been obtained in the previous articles.

As in Art 81, we begin by arguing that if a system of forces $m\omega^2 \cdot PN$ acting along NP be introduced on the respective elements, all these elements, and in consequence the whole mass of the liquid, are reduced to rest Thus this system of forces together with gravity will give rise to the same fluid thrust and the same pressure at every point as in the actual case of rotation

Taking Oy vertically upwards, we see that under these conditions $Y = -g$, $X = \omega^2 x$, $Z = \omega^2 z$, the last two being components of $\omega^2 \cdot PN$

Substituting these values in (4) of Art 73, we have

$$dp = \varrho\,(\omega^2 x\,dx + \omega^2 z\,dz - g\,dy)$$

$\therefore$ integrating, $p = \varrho\,[\tfrac{1}{2}\omega^2 (x^2 + z^2) - gy] + C$,

or
$$y = \frac{\omega^2}{2g}(x^2 + z^2) + \frac{C - p}{g\varrho}, \qquad \ldots\ldots(13)$$

a form which can easily be reconciled with the results of the preceding articles

85. Illustrative examples. In solving some examples it becomes necessary to determine the volume of a segment of a paraboloid bounded by a plane perpendicular to its axis It is therefore helpful to remember that this volume $= \frac{1}{2}$ (area of the base $\times$ altitude)

Ex. 1 An open cylindrical vessel, full of water, is made to rotate with uniform angular velocity ω about its axis, which is kept vertical Find the amount of water that escapes and the pressure on the base

Let the radius of the base be a and the height be h There will be two cases the angular velocity may be such that the vertex of the free surface may be (1) above the base of the vessel, as in the right-hand figure, (2) below the base, as in the left-hand figure

Any point on the rim is on the free surface, because the liquid as it escapes from the vessel does so across the rim ,

therefore there will always be some liquid at the rim, although there is no liquid in its plane The rim will,

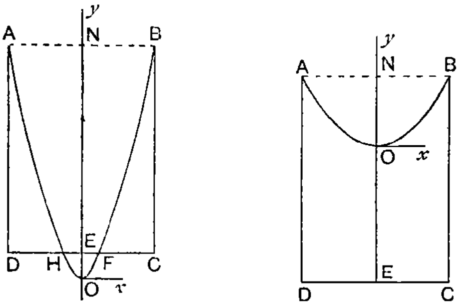


FIG 55.

therefore be on the common surface of the liquid and the air.

Let AOB be the free surface, its equation is

$$y = \frac{\omega^2}{2g} x^2,$$

referred to O as the origin The coordinates of B are (a, ON)

$$\therefore \ ON = \frac{\omega^2}{2g} a^2, \qquad \qquad (1)$$

giving the depth of the vertex below the top of the vessel In case (1) $ON < h$,

$$\therefore \ \frac{\omega^2 a^2}{2g} < h \text{ or } \omega < \frac{1}{a} \sqrt{2gh}.$$

It appears from (1) that ON increases as ω increases Therefore when ω is less than $\frac{1}{a}\sqrt{2gh}$, O is above CD, when ω is equal to this value, O is on the base so that the base becomes just visible Finally when ω exceeds this, O is below CD and a circular part, HF, of the base becomes exposed as in the left-hand figure.

In case (1), vol. of the liquid that has escaped

$$=\text{vol. of the paraboloid } AOB \text{ (rt.-hand fig.)}$$

$$=\tfrac{1}{2}\pi a^2 . ON = \frac{\pi\omega^2 a^4}{4g}.$$

In case (2), this volume

$$=\text{paraboloid } OAB - \text{paraboloid } OHF \quad \ldots\ldots\ldots \text{(ii)}$$

To evaluate the latter, we see that

$$OE = ON - h = \frac{\omega^2 a^2}{2g} - h, \text{ and } OE = \frac{\omega^2}{2g} EF^2,$$

since F is on the free surface

$\therefore$ (ii) becomes $\tfrac{1}{2}\pi a^2 . ON - \tfrac{1}{2}\pi . EF^2 . OE$,

or
$$\frac{\pi\omega^2 a^4}{4g} - \frac{\pi}{} \frac{g}{\omega^2}\left(\frac{\omega^2 a^2}{2g} - h\right)^2,$$

or
$$\pi h\left(a^2 - \frac{gh}{\omega^2}\right)$$

When the base is just visible, the vol. of the liquid that escapes $= \tfrac{1}{2}\pi a^2 h =$ half the volume of the cylinder.

The pressure on the base, by *Note*, Art. 82, neglecting the pressure of the air,

$=$ wt. of the liquid above the base and below the free surface

$$=\left(\pi a^2 h - \frac{\pi\omega^2 a^4}{4g}\right) g\varrho, \text{ in case (1)},$$

or
$$=\frac{\pi g h^2}{\omega^2} g\varrho, \text{ in case (2)}.$$

If the atmospheric pressure be included, we have to add $\Pi . \pi a^2$ to the above expressions.

Ex. 2. A closed cylindrical vessel is just filled with water, and the whole is rotated uniformly about the axis of the vessel, which is kept vertical. Find the pressure on the base.

Let $ABCD$ be the closed vessel and ω the angular velocity, also let $OE = h$ and $OA = a$.

When the whole is rotating uniformly, the surfaces of equal pressure are paraboloids It can easily be seen that, in the same horizontal level, the pressure at the point on the axis OE will be the least (Art 82) Also, on the axis OE, the pressure at a point diminishes as we go up Therefore the pressure at O, the centre of the top, will be less than the pressure at any other point within the cylinder Since the closed cylinder is just filled there will be at least one point where the pressure will be zero, that point is obviously O, since fluid pressure cannot be negative Draw the free surface (of zero pressure), it will be through O, and is represented by KOH in the figure (above the cylinder)

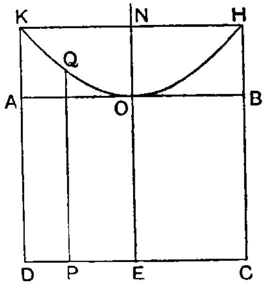


FIG 56

Since pressure at P = wt of column PQ, from (9), Art 82,

$\therefore$ pressure on the base

$$= \text{wt of volume } KOHCD \text{ of water}$$

$$= g\varrho \text{ (vol of cyl } DCHK - \text{vol. of paraboloid } KOH)$$

$$= g\varrho \left[\pi a^2\left(h + \frac{\omega^2 a^2}{2g}\right) - \frac{\pi\omega^2 a^4}{4g}\right]$$

$$= \pi a^2 g\varrho\left(h + \frac{\omega^2 a^2}{4g}\right),$$

since $ON = \dfrac{\omega^2 a^2}{2g}$, as in Ex 1

Ex 3 A thin uniform tube bent into the form of a circle of radius a contains a liquid which occupies an arc subtending an angle 2α at the centre This is rotated with uniform angular velocity ω about a diameter which is kept

vertical Show that the liquids on the two sides of the vertical diameter will not separate if ω does not exceed $\sec\frac{\alpha}{2}\sqrt{\frac{g}{a}}$

Let OD be the vertical diameter and A, B the two extremities of the liquid, so that $\angle OCA = \angle OCB = \alpha$ It is clear that the free surface will always pass through A and B Its equation is $y = \frac{\omega^2}{2g}x^2$, referred to its vertex as the origin and the vertical line as the y-axis Now the abscissa of $A = AN = a \sin OCA$

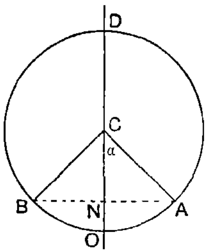


FIG 57

its ordinate or the depth of the vertex of the free surface below N

$$= \frac{\omega^2}{2g} a^2 \sin^2 OCA \quad . \quad \ldots\ldots(1)$$

This shows that the vertex descends as ω increases Now so long as the arc AB of the circle is entirely below the free surface the liquid in this arc remains intact If, however, some portion of this arc be above the free surface (of zero pressure), there cannot exist any liquid in that part, for then the pressure there would be less than zero or be negative It follows, therefore, that so long as the vertex of the free surface does not go below O the liquids in the arcs OA, OB will not separate Hence for the limiting case, O is the vertex of the parabola $y = \frac{\omega^2}{2g}x^2$

$$\therefore \frac{\omega^2}{2g} a^2 \sin^2 OCA = ON,$$

or
$$\omega^2 a^2 \sin^2\alpha = 2ga(1 - \cos\alpha),$$

or
$$\omega^2 = \frac{g}{a}\sec^2\frac{\alpha}{2},$$

whence the result

Note. When the parts separate, the free surface will cut the circle in two more points A' and B' (say) besides A and B, the $\angle ACA' = \angle BCB' = \alpha$ in that case The angle OCA will obviously be greater than α But (1) would still hold

Ex 4. A straight tube AB of thin uniform bore is closed at the lower end A and filled with water The length of the tube is l and it rotates with constant angular velocity ω about a vertical line through A, to which the tube is inclined at a constant angle α Find whether the liquid escapes or not, in the former case find how much of the liquid escapes Also determine the value of ω so that (i) no liquid may escape, (ii) all liquid may escape Atmospheric pressure is to be neglected

Let AB denote the tube and AN the line about which it rotates The water that escapes does so across the open end, therefore there must always be a tiny drop of water at the point B of the open end of the tube farthest removed from the vertical axis We express this fact by saying that B is on the free surface of the water in the tube Let this surface be given by

$$y = \frac{\omega^2}{2g} x^2, \qquad \dots(1)$$

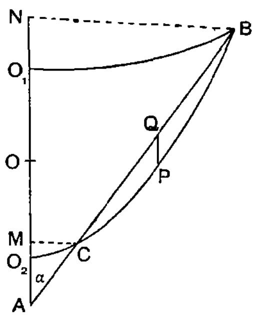


FIG 58

referred to its vertex as the origin and AN as the y-axis

There will be two cases according as the free surface (1) does not cut the tube again below B or (2) cuts AB again at C

As the free surface is one of zero pressure no liquid can remain above this surface Therefore in the first case no liquid escapes, whilst in the second the length CB of the tube is emptied

Case 1. Let O_1B denote the free surface ; then

$$O_1N = \frac{\omega^2}{2g} BN^2 = \frac{\omega^2}{2g} l^2 \sin^2\alpha, \quad \ldots \ldots \quad \ldots \text{(ii)}$$

from (i) as $BN = AB \sin \alpha$

(ii) shows that O_1N increases with ω Let the value of ω ($=\omega_1$, say) be such that the parabola touches AB at B, and let O be its vertex in that case No water escapes from the tube since the whole of it still lies below the parabola AN is now the subtangent , therefore O must be the middle point of AN From (i) we have, in this case,

$$ON = \tfrac{1}{2}AN = \tfrac{1}{2}l \cos \alpha = \frac{\omega_1^2}{2g} l^2 \sin^2\alpha,$$

$$\therefore \omega_1^2 = \frac{g}{l} \frac{\cos \alpha}{\sin^2\alpha} \quad \ldots \quad \ldots \quad \ldots \ldots \text{(iii)}$$

If the angular velocity exceeds ω_1, the vertex of the parabola, viz O_2, will be lower than O, and so this curve will cut AB again at C Hence if $\omega \leqslant \omega_1$, no liquid escapes , if $\omega > \omega_1$ a length BC of the liquid escapes

Case 2 Let $AC = z$, then

$$CM = z \sin \alpha, \quad AM = z \cos \alpha$$

Since B and C are both on the free surface given by (i),

$$O_2N = \frac{\omega^2}{2g} l^2 \sin^2\alpha \text{ and } O_2M = \frac{\omega^2}{2g} z^2 \sin^2\alpha$$

$$\therefore MN = \frac{\omega^2}{2g}(l^2 - z^2) \sin^2\alpha$$

But $MN = AN - AM = (l - z) \cos \alpha$ Therefore equating both values and removing the common factor $l - z$, we get

$$\frac{\omega^2}{2g}(l + z) \sin^2\alpha = \cos \alpha$$

$$\therefore CB = l - z = 2l - \frac{2g}{\omega^2} \frac{\cos \alpha}{\sin^2\alpha} = 2l\left(1 - \frac{\omega_1^2}{\omega^2}\right), \quad \ldots \ldots \text{(iv)}$$

giving the length of the liquid that escapes

Lastly, if the vertex of the parabola comes down to A for a value ω_2 of the angular velocity, the whole of AB lies above it and so all the liquid escapes In this case,

$$AN \text{ or } l \cos\alpha = \frac{\omega_2^2}{2g} l^2 \sin^2\alpha,$$

or $$\omega_2^2 = \frac{2g}{l} \frac{\cos\alpha}{\sin^2\alpha} = 2\omega_1^2 \quad \ldots\text{(v)}$$

Ex 5 If the pressure on the free surface be taken as Π, find the greatest angular velocity with which the tube (in the last example) can be rotated so that no water can escape , the density of water is ϱ

We shall, for the sake of simplicity, assume the tube to be sufficiently long so that the free surface (in the required case) cuts AB at an intermediate point C as in Fig 58. The same argument would apply if this does not happen and the point C lies on BA produced , the results would be the same in each case It would be a good exercise for the student to draw the figure in the latter case and verify the successive steps It may be noted that the middle point of the chord CB of the parabola always lies between A and B

Let BCO_2 (Fig 58) be the free surface when the angular velocity is ω The pressure at a point Q (lying between B and C) will be $\Pi - g\varrho \; PQ$, from (9) of Art 82, since Q is above P Therefore the pressure will be the least at the middle point of BC because for this point PQ is maximum Thus we see that BC will be full of water so long as the pressure at its middle point is not less than zero (*i e* negative) The maximum angular velocity will therefore be when the pressure at this point, Q, is zero

$$\therefore \; PQ \text{ must be} = \frac{\Pi}{g\varrho} \quad \ldots\ldots\text{(vi)}$$

Taking O_2 as origin, the parabola CPB is given by $y = \dfrac{\omega^2}{2g} x^2$, where ω represents the maximum angular velocity required.

$$\therefore \quad y_Q = \tfrac{1}{2}(y_B + y_C) = \frac{\omega^2}{4g}(l^2 + z^2)\sin^2\alpha.$$

The abscissa of P

$$= \text{the abscissa of } Q = \tfrac{1}{2}(BN + CM) = \tfrac{1}{2}(l + z)\sin\alpha$$

$$\therefore \quad y_P = \frac{\omega^2}{8g}(l + z)^2 \sin^2\alpha.$$

$$\therefore \quad PQ = y_Q - y_P = \frac{\omega^2}{8g}(l - z)^2 \sin^2\alpha, \text{ on simplification,}$$

$$= \frac{\omega^2}{2g}\sin^2\alpha\left[l - \frac{g}{\omega^2}\frac{\cos\alpha}{\sin^2\alpha}\right]^2,$$

from (iv) of last example

$\therefore$ equating this expression with (vi) and simplifying,

$$l - \frac{g}{\omega^2}\frac{\cos\alpha}{\sin^2\alpha} = \frac{1}{\omega\sin\alpha}\sqrt{\frac{2l}{\varrho}} \qquad \text{(vii)}$$

(vii) gives ω if l is given, conversely, it would give l if ω were given, i e it would give the length of the longest tube which can rotate with angular velocity ω without any liquid escaping.

Ex 6 A solid in the form of a right circular cylinder floats in water contained in a cylindrical vessel, with its axis vertical and coincident with that of the vessel The whole is then made to rotate without relative motion about the common axis with uniform angular velocity ω Show that the solid sinks (in space) through a distance $\dfrac{\omega^2}{4g}(b^2 - a^2)$, b and a denoting the radii of the bases of the vessel and the solid respectively

The right-hand figure shows the position of the solid when the whole system is at rest, and the left-hand figure

gives its position during rotation $D'C'$ is the free surface in the former and DOC in the latter

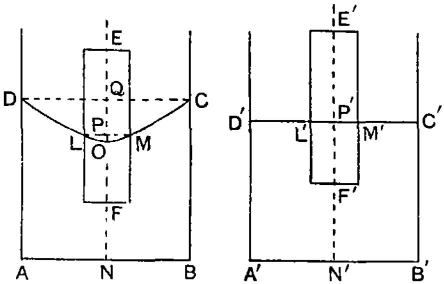


FIG. 59.

Since the solid has no vertical motion (even in the state of rotation), it follows that its weight balances the fluid thrust on it (= the weight of the liquid displaced). In the second case the free surface (a paraboloid) intersects the solid along a circle which is denoted by LM.

∴ the vol of the liquid displaced = cyl FLM − paraboloid LOM, in the second case; also = cyl. $F'L'M'$ in the first As the weights of the liquid displaced must be equal,

$$\therefore \text{ cyl } F'L'M' = \text{cyl } FLM - \text{par } LOM. \quad \ldots\ldots(\text{i})$$

Also the volume of water in the vessel remains the same.

$$\therefore \text{ cyl } A'B'C'D - \text{cyl. } F'L'M'$$
$$= \text{cyl } ABCD - \text{par. } DOC - (\text{cyl } FLM - \text{par. } LOM)$$

∴ Adding (i) we get

$$\text{cyl } A'B'C'D' = \text{cyl } ABCD - \text{par. } DOC. \quad \ldots\ldots(\text{ii})$$

Let $A'D' = h'$, $P'F' = l'$, $AD = h$, $PF = l$ The free surface in the left-hand figure has for its equation

$$y = \frac{\omega^2}{2g} x^2.$$

$$\therefore OP = \frac{\omega^2 a^2}{2g}, \quad OQ = \frac{\omega^2 b^2}{2g}, \quad \ldots\ldots \ldots\ldots(\text{iii})$$

as $PM=a$ and $QC=b$. Therefore (i) and (ii) become

$$\pi a^2 l' = \pi a^2 l - \tfrac{1}{2}\pi a^2 \frac{\omega^2 a^2}{2g},$$

and
$$\pi b^2 h' = \pi b^2 h - \tfrac{1}{2}\pi b^2 \cdot \frac{\omega^2 b^2}{2g}.$$

Or
$$l' = l - \frac{\omega^2 a^2}{4g}, \quad . \quad . \quad . \quad . \text{(iv)}$$

and
$$h' = h - \frac{\omega^2 b^2}{4g} \quad . \quad \dots \; \dots \text{(v)}$$

Now, $NF = h - l - PQ = h - l - (OQ - OP)$

$$= h - l - \frac{\omega^2}{2g}(b^2 - a^2), \text{ from (iii).}$$

And $N'F' = h' - l' = h - l - \dfrac{\omega^2}{4g}(b^2 - a^2)$, from (iv) and (v),

$$= NF + \frac{\omega^2}{4g}(b^2 - a^2).$$

This shows that the base of the floating solid was farther from the base of the vessel when at rest than when in rotation, whence the result

EXAMPLES. 11.

1. A cylindrical vessel is half-filled with liquid. It is then rotated uniformly about the (vertical) axis of the vessel so that the liquid just reaches the rim. Neglecting the atmospheric pressure, prove that the pressure at the centre of the base is zero, and that the pressure at any other point of the base is proportional to the square of its distance from the centre. Hence determine the resultant fluid pressure on the base

2. A closed cylindrical vessel is just filled with water and is made to rotate about its axis (which is vertical) with uniform angular velocity; find its value so that the pressure on the base may be equal to twice the weight of the liquid contained in the vessel.

3. A closed conical vessel, of height h and semi-vertical angle α, is just filled with liquid and placed with its axis vertical and base downwards. The whole is then rotated

about the axis with uniform angular velocity ω Compare the pressure on the base with the weight of the contained liquid

4. A cylindrical vessel, open at the top and originally full of water, rotates uniformly about its axis With what angular velocity should the whole revolve so that only one-fourth of the original quantity of water remains in the vessel ?

5. A closed vessel in the shape of a right circular cone is full of water and placed with its axis vertical and vertex downwards It is then made to rotate with uniform angular velocity ω ; show that the pressure at a point of its plane face at a distance x from the axis is $\frac{1}{2}\rho\omega^2x^2$ and at a point on the curved surface at the same distance from the axis is

$$\tfrac{1}{2}\rho\omega^2x^2 + g\rho(a-x)\cot\alpha,$$

where a is the radius of the base and 2α the vertical angle of the cone

6 When a cylinder, open at the top and half-full of liquid, revolves with angular velocity ω about its axis (which is vertical), the liquid just reaches the upper rim Show that the angular velocity in order that $\frac{1}{n}$th of the liquid may remain in the vessel is $\omega\sqrt{n}$

7. A right circular cylinder of radius a is floating freely in a liquid at rest with its axis vertical, then the whole is made to rotate without relative motion about the axis of the cylinder with constant angular velocity ω Show that an extra amount of the surface of the cylinder is now wetted, and find its magnitude.

8. A liquid occupies a portion of a thin circular tube of uniform bore, which subtends an angle 210° at the centre The tube rotates about a tangent (which is kept vertical) with angular velocity ω, find its value so that the liquid may just reach the highest point of the tube The radius of the circle is a.

9. A circular tube of fine bore, in the form of a quadrant of a circle, is closed at the lower end, one-third of its length is filled with a liquid When it is rotating uniformly about the radius through the closed end, which is kept vertical, the liquid just rises to the top of the tube Prove that the vertex of the free surface is at a depth $2a$ below the centre of the circle, where a is its radius

10. A conical vessel of height h and vertical angle 60°, has its axis vertical and is half-filled with water. Find the greatest angular velocity with which the whole can rotate about the vertical axis without any water overflowing.

11. A vessel in the shape of a paraboloid of revolution is placed with its axis vertical and the plane of its rim horizontal. Some liquid is poured into it and the whole revolves uniformly about the vertical axis. Show that no liquid can remain in the vessel if the angular velocity exceeds a certain value

12. If, in the last example, a hole be made at the vertex of the vessel, show that no angular velocity can prevent the whole liquid from escaping

13. A thin circular tube of radius a contains a filament of liquid subtending an angle 2α at the centre, and rotates uniformly with angular velocity ω about the vertical diameter so that the filament divides into two equal parts Prove that the middle points of these halves subtend an angle

$$2 \cos^{-1}\left[g \sec \frac{\alpha}{2} \Big/ a\omega^2\right]$$

at the centre.

14. A tube of uniform small section is in the form of three sides of a square of which the middle side is horizontal, the other two being vertical, it is open at both ends The tube is full of water and revolves about a vertical axis bisecting the horizontal side. Prove that no liquid escapes unless the angular velocity, ω, is greater than $\sqrt{8g/a}$, in which case the amount of liquid that escapes would fill a length $a\sqrt{1 - \frac{8g}{a\omega^2}}$ of the tube, a being the length of a side

15. A tube ABC of fine uniform bore and length $a+b$ is in the shape of a right angle, the arm BC being vertical and of length b. The end A (which is lower) is closed and the tube is filled with liquid. It is then rotated with uniform angular velocity ω about the vertical through A so that some liquid escapes through the open end C. Find how much liquid remains in the tube and its position.

16. If the atmospheric pressure P be taken into consideration, show that no liquid will escape from the tube of the last example, if $a^2\omega^2$ does not exceed $2bg + \frac{2P}{\varrho}$, ϱ denoting the density of the liquid.

---

17 A circular tube of thin uniform bore and radius a is filled with liquid and is made to rotate about its vertical diameter with angular velocity ω Find the pressure at any point of the tube, and show that it is maximum at a depth g/ω^2 below the centre if $\omega > \sqrt{g/a}$, otherwise it is greatest at the lowest point of the tube

18. In the previous example, where should a hole be made so that all the liquid in the tube may escape?

19. A spherical vessel, of radius a, contains a quantity of water whose volume is to the capacity of the vessel as $n^3 : 1$. Prove that no water can escape through a small hole made at the lowest point if the whole revolves about the vertical diameter with an angular velocity which is not less than $\sqrt{\dfrac{g}{a(1-n)}}$.

20. A hemispherical bowl, of radius $2a$, containing water rotates uniformly with an angular velocity $\sqrt{\dfrac{3g}{7a}}$ about its axis which is vertical. A sphere of radius a is floating in the water and rotating with it; the axis of the vessel is coincident with the vertical diameter of the sphere, and the sphere remains with its lowest point in contact with the vessel without exerting any pressure on it. If the free surface, at this instant, passes through the rim of the bowl, show that the sp. gr. of the sphere is $\frac{128}{189}$.

21. A cylinder, of radius r and height h, floats with its axis vertical in a given mass of a liquid whose density is m times that of the cylinder. The whole revolves without relative motion about the axis of the cylinder so that its rim is just immersed. Prove that the angular velocity ω is given by

$$mr^2\omega^2 = 4gh(m-1).$$

22. A solid, in the form of a right circular cone of semi-vertical angle α, floats with vertex downwards and axis vertical in water contained in a cylindrical vessel whose axis coincides with that of the cone. The surface of the water meets the cone in a circle of diameter a. If the whole is now made to revolve about the common axis so that the water meets the cone in a circle of radius a, find the value of the angular velocity.

23. A vessel consists of a hemispherical base surmounted by a cylinder. Liquid is poured into this vessel and is rotated with uniform angular velocity ω about the axis which is vertical. Prove that, on the surface of the hemisphere, the pressure would be maximum at a point distant g/ω^2 from the plane base of the hemisphere provided that $\omega^2 r > g$. The atmospheric pressure is neglected and the radius of the hemisphere is r.

24. Liquid is rotating in a cylinder of radius a, whose bottom is closed by a conical surface of vertical angle 90°, the vertex of the cone being downwards Find the position of the point or the surface of the cone where the pressure is minimum if the angular velocity $> \sqrt{g/a}$

25. An elliptic tube, half-full of liquid, revolves about a tangent which is kept vertical with uniform angular velocity ω. Prove that the diameter joining the free surfaces of the liquid makes with the horizontal an angle $\cot^{-1} \frac{g}{\omega^2 p}$, where p is the length of the perpendicular from the centre of the ellipse on the tangent How is the result modified if the ellipse revolves about a vertical line in its plane, which does not intersect the tube ?

26. An open cylinder containing liquid is revolving about a vertical axis with angular velocity $\sqrt{\frac{ng}{a}}$, where $n > 1$ and a is the radius of the base. A sphere of radius a is immersed so as to be just supported by the liquid alone, and it rests with its lowest point at the vertex of the free surface (where the pressure is to be taken to be zero). If the density of the liquid is m times that of the sphere, prove that

$$m^{-\frac{1}{3}} + n^{-1} = 1$$

27. A heavy homogeneous liquid is revolving without relative motion about a vertical axis with uniform angular velocity $\sqrt{\frac{mg}{b}}$ Show that the centre of pressure of a vertical rectangle of sides $2a$, $2b$ (taken so that the upper horizontal side of length $2b$, is bisected at the free surface), and revolving with the liquid, is at a depth

$$\frac{mb + 8a}{mb + 6a} \cdot a$$

86. There are a few problems which can be treated in a manner similar to that employed in rotating liquids

For example, suppose a vessel containing some liquid be moving on a horizontal plane with uniform acceleration f, and let the liquid be moving without relative motion along with the vessel. To find the free surface of the liquid.

Let the whole be moving in the direction from left to right as indicated by the arrow with constant acceleration f The forces acting on a particle of liquid of mass m are (1) its weight mg and (2) the resultant fluid thrust mF, whose direction is at present unknown These give to the particle the acceleration f, therefore they would be balanced by a force mf acting from right to left. Considering the equilibrium of these three forces, we get

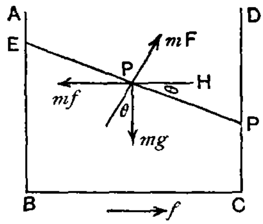


FIG 60.

$$mg = mF \cos\theta \quad \text{and} \quad mf = mF \sin\theta,$$

where θ is the angle the direction of the force mF makes with the vertical

$$\therefore \tan\theta = \frac{f}{g} = \text{constant} \quad \ldots \ldots \quad (14)$$

Thus mF is in a definite direction, since it is always at right angles to the surfaces of equal pressure, it follows that the latter must be planes inclined at angle θ to the horizontal (Art 74) Therefore the free surface, EP, would be a plane inclined at $\tan^{-1}\dfrac{f}{g}$ to the horizontal, so that the volume $EBCP$ would be equal to the given volume of the liquid

In a similar manner we can solve the problem of the vessel, containing some liquid, which moves along an inclined plane with acceleration f It is interesting to note that if the vessel moves down the plane freely under gravity (without friction) the free surface is parallel to the inclined plane

CHAPTER IX

GASES PRESSURE OF ATMOSPHERE

87. Gas is one of the three forms in which substances present themselves to us We have defined it, in the first chapter, as a substance which has no definite size or shape of its own, and also as an elastic or compressible fluid We have often been speaking of the density of atmosphere and the pressure exerted by it in the previous chapters We shall now put these notions on experimental basis

When we speak of a thing as a substance or we speak of its density we imply that the thing has mass (and, consequently, weight) For solids and liquids there is never any doubt as to their possessing weight That the gases also have weight can be seen from the following experiment. Take a flask full of air (or any other gas) and weigh it. Next remove the air from inside by means of an exhaust pump, close the flask tightly and weigh it again If weighings are made very carefully it will be found that the weight in the second case is less than that in the first . this demonstrates that a given volume of a gas has some weight (or mass), although it is so small that we are prone to overlook this fact But it should be noted that the density of a given quantity of gas does not remain constant , it changes with the change of temperature and pressure. In the case of solids and liquids, temperature has some slight effect on their densities, but in gases the effect is much more marked. The following table giving the sp gr of a few gases shows

how small is the sp gr of a gas in ordinary circumstances Water is taken as the standard substance

Hydrogen	-	·000089.	Air	-	·001293.
Nitrogen	-	001256	Oxygen	-	·001430
Carbon dioxide	-	·001977	Steam	-	000802

Hydrogen is the lightest gas known, for this reason it is chosen as the standard substance for measuring the sp gr of gases. Thus the values of the sp gr of the above gases will be 1, 14, 22, 14 5, 16, 9 respectively

Air at 0° C and 76 cm pressure is sometimes taken as the standard for comparison of the densities of gases

88. That gases exert pressure can be shown by a large number of simple experiments, a few of which are given below.

If a toy balloon, generally made of rubber, be filled with a gas, the envelope expands owing to the pressure exerted by the gas (within it) on its surface

If the air within a tube be sucked out at one end whilst the other end is closed by a flat piece of paper pressing against it, it will be noticed that the paper remains of itself against the end without any apparent force being applied to keep it in position The pressure exerted by the external air keeps the paper pressed against the tube

Magdeburg's experiment Two hollow hemispheres made of metal fit tightly together so as to form an air-tight hollow sphere The air within the sphere is then removed by a suction pump It is found that very great force is now necessary to separate the two hemispheres, the reason being that the pressure of the external air (which was compensated by the pressure of the air within the sphere before exhaustion) adds considerably to the resistance against the separation of the parts

If a glass beaker is held mouth downwards on the surface of water and then pressed down, the air in the beaker will

be seen to have forced down the level of water within the glass

89. We have seen that gas has weight, so a given mass of gas at rest under gravity will necessarily exert pressure the magnitude of which (over any small area) will be equal to the weight of the column of the gas above it (cf Art 9) The atmospheric pressure is an instance of this type, but we do not feel the pressure on our bodies since we are accustomed to it If we happen to be at a place where the pressure is either appreciably less or appreciably more, we would feel some inconvenience which is entirely due to the change in pressure At great heights above the surface of the earth we always experience difficulty with breathing and other attendant discomforts this is mainly because the pressure of the air there is considerably less than that at the surface of the earth

Since the weight of air (or a gas) per unit volume is very small, the pressure of the air remains practically constant within a vertical distance of a few feet For this reason we generally consider *the pressure of a finite enclosed volume of a gas to be the same at every point within its volume*, we speak of this constant pressure as the pressure of the given volume of the gas

Again, a gas, like other fluids, exerts pressure normally to the portion of the surface with which it is in contact If a solid be surrounded by a gas, the resultant fluid thrust exerted by the gas on the solid is upwards and equal to the weight of an equal volume of the gas (Art 40) But on a liquid the effect is different, since the liquid presents its upper surface only to gaseous pressure, the latter tends to press the liquid down This fact is taken advantage of in measuring the atmospheric pressure Mercury is generally chosen as the liquid employed for this purpose, because (1) its sp. gr is large, viz 13 596, (2) it can be obtained

pure, (3) it does not easily evaporate at ordinary temperatures

The instrument by means of which the atmospheric pressure is measured is called a *barometer*

90. A *mercury barometer* in the simplest case consists of a long tube CE of uniform bore (not very small) and of length 36 inches nearly, one end E of this tube is closed It is filled with mercury, with the open end closed by the thumb (or by any other means) it is inverted and put, with this end below the level of mercury contained in a cylindrical cistern AFB, in a vertical position The thumb is then removed and the end left free Some mercury will descend from the vertical tube into the cistern, but a length CH remains in the tube

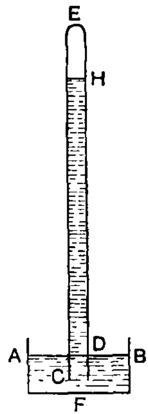


FIG 61

The space above the mercury level, H, in the tube does not practically contain anything, *i e* it is *in vacuo*, this space is known as the *Torricelli's vacuum* So the pressure at the level H is zero and that at the level ADB is equal to the weight of the column DH of mercury This column is sustained in this position by the atmospheric pressure acting on the surface AB of the mercury in the cistern Therefore the atmospheric pressure

$$= \text{the pressure at the level } ADB$$

$$= \text{weight of the column } DH \text{ of mercury}$$

$$= g\rho h, \qquad \ldots \qquad (1)$$

where $h = DH$, ρ = density of mercury and g the acceleration due to gravity The height h of the mercury-level in the vertical tube above that in the cistern is called the *height of the barometer*

91. The barometer just described is sometimes spoken of as the *cistern* barometer. As the atmospheric pressure increases or decreases, the height DH will also increase or decrease, *i e* the quantity of the mercury in the tube and consequently the quantity in the cistern changes This means that the level, ADB, of the mercury in the cistern moves up or down Since the barometric height is always measured above the mercury-level in the cistern, to avoid the necessity of measuring from different levels means are devised by which the mercury-level in the cistern can be brought to a definite position, such as a flexible bottom to the cistern, whose capacity can be increased or decreased by turning a screw attached to its base The height can then be measured by a scale (fixed in position) above the definite level mentioned before *Fortin's barometer* is one of such type

There is another kind of mercury barometer known as the *siphon* barometer, it consists of a U-tube, one arm of which is long and narrow and the other arm short and wide (Fig 62) The length of the longer arm, CE, is a little more than 36 inches and is uniform in cross-section. This arm is held in a slanting position and filled with mercury, it is then brought to the vertical position, when some mercury descends and the mercury-levels in the two arms stand at H and AB The height of the level H above AB or D gives the pressure of the atmosphere ; the portion HE is vacuum as in the cistern barometer

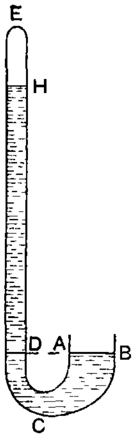


Fig 62

Note. Water may also be used in the construction of a barometer instead of mercury, or in fact any other liquid, but then the tube CE should be much longer, the length in the case of water being about 35 feet There

is another disadvantage, viz the space HE would not be so perfect a vacuum as in the case of mercury, because some water evaporates from the level H, the water-vapour formed occupies this space and exerts a downward pressure on the level H of water The level H is therefore depressed, and thus the height DH would not give the true barometric height

92. To measure h [formula (1)], we want a scale either fixed to the tube or etched on its surface In Fortin's barometer the scale is in inches or centimetres with convenient subdivisions, the zero of the scale being at the definite level of the cistern But in other barometers allowances must be made for the capacity of the cistern (or the wider arm of the siphon barometer) That is to say, the graduations of the scale should not be made at distances of one inch (or centimetre), but at some other equal intervals, so that when the level rises or falls through m divisions we shall be able to say at once that the barometric height has risen or fallen through m inches (or centimetres) To find the value of this unit in terms of an inch (or centimetre) we proceed as follows

Let a and A denote the cross-sections of the tube and the cistern When mercury in the tube falls through one division ($=x$ inches, say), the volume of mercury leaving the tube and entering the cistern $=ax$ The level in the cistern will therefore be raised by $\frac{ax}{A-a}$, since the area of the surface of mercury in it is $A-a$ Hence the original difference between the levels is diminished by

$$x+\frac{ax}{A-a}, \text{ or by } \frac{A}{A-a}\ x$$

But this has been assumed to be 1

$$\therefore\ x=\frac{A-a}{A}=1-\frac{a}{A} \quad \ldots \quad (2)$$

In siphon barometer, if A denote the cross-section of the wider arm, supposed uniform near the level AB, the fall of mercury in the tube CE through one division (or x inches) is accompanied by a rise in the level in the other by $\frac{ax}{A}$. Therefore the diminution of the barometric height is $x+\frac{ax}{A}$, or $\frac{A+a}{A}x$ Hence

$$x=\frac{A}{A+a} \quad \ldots\ldots\ldots\ldots (3)$$

If, however, the barometer be not graduated according to the new units given by (2) or (3), but in actual inches (or centimetres), then a *rise or fall as given by the scale must be multiplied by* $\frac{A}{A-a}$ $\left(\textit{or by } \frac{A+a}{A} \textit{ in the case of siphon barometer}\right)$ *to obtain the true rise or fall in the barometric height. This correction is known as that due to the capacity of the cistern* ....(4)

93. The scale which is attached to the barometer tube (or etched on it) does not give true readings at all temperatures, since the material of the scale expands or contracts as the temperature rises or falls This necessitates another correction to be made to the reading as given by the scale Let the coefficient of linear expansion of the material of the scale (metal or glass) be α, and suppose that the scale gives true reading at 0° C, *i e* the distance between two divisions on the scale is exactly what it is marked to be when the scale is at 0° C At t° C the distance between the two divisions will be $(1+\alpha t)$ times the distance at 0° C Therefore *the value h' as given by the scale must be multiplied by $(1+\alpha t)$ in order to obtain its true value, h* Or

$$h=h'(1+\alpha t) \quad \ldots\ldots\ldots\ldots (5)$$

Again, the density of mercury (used in the barometer) diminishes with the rise of temperature If γ be the

coefficient of cubical expansion of mercury, ϱ_0 its density at 0° C and ϱ the density at $t°$ C,

$$\varrho = \frac{\varrho_0}{1+\gamma t}, \quad \ldots \ldots \ldots (6)$$

since the volume at $t°$ C $=(1+\gamma t)$ times the volume of the mass at 0° C

∴ Substituting in (1) we get

$$g\varrho h = g\varrho_0 h' \frac{1+\alpha t}{1+\gamma t}$$
$$= g\varrho_0 h' (1+\gamma t)^{-1} (1+\alpha t)$$
$$= g\varrho_0 h'[1-(\gamma-\alpha)t], \quad \ldots \ldots (7)$$

neglecting terms of second and higher degrees in α, γ since they are small

The correction indicated in (7) is known as *that due to change of temperature*

94. In the formula (1) occurs g, or the value of the acceleration due to gravity This has different values for different latitudes, as also at different heights above the earth's surface There will, therefore, be some correction on this account It will be out of place here to go into the discussion regarding the cause and the amount of the change in the value of g The formula on the earth's surface at latitude φ is

$$g_0 = 32{\cdot}17(1 - {\cdot}002644 \cos 2\varphi + {\cdot}000007 \cos^2 2\varphi) \text{ ft/sec}^2 \quad \ldots (8)$$

And that at a height z above the surface of the earth is

$$g = g_0 \frac{a^2}{(a+z)^2}, \quad \ldots \ldots \ldots (9)$$

where a denotes the radius of the earth and g_0 the value of gravity at the earth's surface

Thus we see that all the three factors of the formula (1) require corrections. (2) or (3) and (5) refer to h, (6) to ϱ and (8), (9) refer to g

95. There are two other corrections, one due to capillarity and the other due to the presence of some vapour in the space above mercury. These are generally very small for a mercury barometer The first correction is a constant quantity for each barometer, and can be determined either experimentally or by calculation with the help of the theory of capillary action

96. There is another kind of barometer, viz the *aneroid barometer*, in which no liquid is employed. The principle of this instrument lies in the action of a thin metallic membrane, the pressures on two sides of which are different This action is magnified by mechanical contrivances, thus affording means of recording the excess of pressure on one side of the membrane over the other. The membrane covers a given enclosed quantity of air at some constant pressure, the other side being exposed to the atmospheric air This barometer, however, is not so accurate as the liquid barometers described before

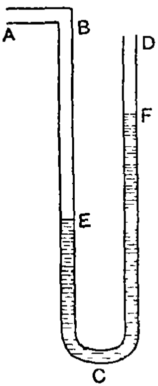


FIG 63

97. *Manometer* is sometimes used to determine the pressure of some enclosed volume of a gas It consists of a tube bent into the form of the adjoining diagram The lower part ECF is filled with mercury (or any other suitable liquid), the horizontal arm BA is connected with the vessel containing the gas, and the end D is open The difference of mercury levels in the two arms is then measured, let it be d Then the pressure of the gas

= the pressure at the level E

= the pressure at the level F + weight of column, d, of

the liquid according as E is below or above F; the pressure at F is that due to atmosphere, whose value can be found by means of a barometer

98. Boyle's law. *The product of the pressure and the volume of a given mass of a gas is constant provided that the temperature is unchanged.*

This important law, also known as Mariotte's law, is based on experimental facts That the volume of a given quantity of gas depends on its pressure (Art. 89) can be seen from the last experiment described in Art 88. In the beginning the whole of the beaker was occupied by air at atmospheric pressure Afterwards the pressure of the enclosed air became equal to the value of the pressure at the water-level inside the beaker, which was greater than the atmospheric pressure, and the volume of the enclosed air was less than before

The apparatus by which Boyle's law can be verified is shown in the figure It consists of a glass tube AB of uniform cross-section closed at the upper end and fixed to a vertical stand, this tube contains some air near its upper end It is connected by means of the rubber tubing C to another tube DE (open at the top), which can move up and down a vertical rod R fixed to the stand, and which can be fixed in any position by means of the clamp F The tube C as well as the lower parts of the tubes AB and DE contain mercury.

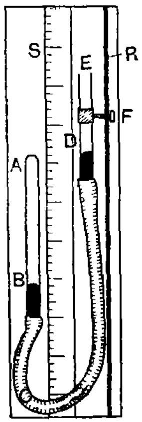


FIG 64.

Atmospheric pressure is first noted with a mercury barometer, let the barometric height be h. The movable tube is fixed in some position, and the height of the mercury level in it above that in the tube AB is measured by means of the

scale S fixed to the stand. Let this be h_1 and the length AB of air be a_1 The tube DE is then moved to some other position and the readings taken again; let a_2 denote the new length of air in the fixed tube and h_2 the difference of the mercury levels

Pressure of the air in AB in the first position $= g\varrho(h+h_1)$, and volume of the air in AB in the first position $= a_1 a$, where ϱ is the density of mercury and a the cross-section of the tube AB

Similarly, the pressure of the air in AB and its volume in the second position are $g\varrho(h+h_2)$ and $a_2 a$ respectively. If Boyle's law is true, we must have

$$g\varrho(h+h_1)a_1 a = g\varrho(h+h_2)a_2 a,$$

or

$$(h+h_1)a_1 = (h+h_2)a_2 \quad \ldots. \quad \ldots \quad .. \qquad (10)$$

This is actually verified by experimental observations

99. A gas which obeys Boyle's law is called a *perfect gas*.

If the experiment described above be carried out very carefully it will be found that the equation (10) is not accurately satisfied The defection from this law for air, oxygen, nitrogen, hydrogen and some other gases is, however, very small, they may therefore be taken as perfect gases under ordinary conditions On the other hand, gases like carbon dioxide, water-vapour, etc, do not behave quite in accordance with this law It has been found that any gas could be liquefied by subjecting it to suitably high pressure provided that its temperature is below a definite value which is called the *critical temperature* for that particular gas When the gas is at a temperature near its critical value or below it, the gas is found not to obey Boyle's law at all At temperatures considerably higher than the critical the gas behaves almost like a perfect gas, and the higher the temperature the more perfect the agreement In the case of air, etc. the ordinary temperatures are far removed from their critical temperatures.

Unless otherwise stated the gas under discussion will be assumed to be a perfect gas.

100. Let m denote the given mass of a gas, and p, v and ϱ denote its pressure, volume and density; let p', v', ϱ' denote the corresponding quantities when the pressure changes to p', the temperature being the same in the two cases Boyle's law then gives

$$pv = p'v' = k_1 \text{ (say)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where k_1 is some constant But $\varrho = \frac{m}{v}$ and $\varrho' = \frac{m}{v'}$,

$$\therefore \; p \frac{m}{\varrho} = p' \frac{m}{\varrho'}, \text{ or } \frac{p}{\varrho} = \frac{p'}{\varrho'}$$

It shows that *the pressure of a gas is proportional to its density provided its temperature is unaltered* That is

$$p = k\varrho, \quad . \quad . \quad . \quad . \quad (12)$$

where k is another constant

The equation (12) may be taken as another mode of stating Boyle's law

101. Two constants k_1 and k are involved in the two different statements of Boyle's law, viz in (11) and (12). *The constant k depends only on the nature of the gas and is independent of its mass, whereas k_1 depends both on the nature and the mass* The following experiments demonstrate the truth of this fact

(1) Let a given mass of a gas be enclosed in a cylindrical vessel into which fits a smooth airtight piston, and let a constant inward force, F, be applied to the piston The object of this is to subject the enclosed gas to a constant pressure, p say To see this, let a be the area of the piston, then pa is the force with which the gas within tends to push the piston This is balanced by F, the pressure on the outer side, and the component of the weight of the

piston in this direction Therefore pa = constant, or p is constant.

Note the volume V of the gas. Next, let some more gas of the same kind be introduced into the vessel. It is found that the piston moves outwards so that the new mass now occupies a greater volume (at the same pressure p) than before. The new volume V' is noted. As

$$V' > V, \quad pV' > pV,$$

or $$k_1' > k_1, \quad \ldots \quad (13)$$

where k_1', k_1 denote the values of the constant of (11) for the new and the original quantities of the gas

(2) Let some air be enclosed in a space, for example in the cylinder of the previous experiment, let its pressure be p and the atmospheric pressure be Π Let the volumes of equal masses of the air in the vessel and of the atmospheric air be V, V' respectively Now, if the former, by suitably changing the magnitude of F, be allowed to attain the pressure Π, it will be seen that its volume changes from V to V', it is right that this should be so, because the former quantity of air when it attains the pressure Π will be indistinguishable from the atmospheric air Applying Boyle's law to this mass, we get

$$pV = \Pi V',$$

and since the mass is the same, $\dfrac{p}{\varrho} = \dfrac{\Pi}{\varrho'}$,

or $$k = k', \quad \ldots \quad (14)$$

where ϱ, ϱ' are the densities and k, k' the constants for the enclosed air and the atmospheric air

From (12), we have

$$p = k\varrho = \frac{km}{v},$$

$$\therefore\ pv = k_1 = km, \quad \ldots \quad (15)$$

showing again that k_1 depends on mass.

102. Charles' law. *If the pressure of a given quantity of gas be constant, its volume increases by* ·003665 (*or* $\frac{1}{273}$ *nearly*) *of the volume of the gas at* 0° *C*., *for every rise of* 1° *C. in its temperature*.

This law is also attributed to Gay-Lussac or to Dalton. Like Boyle's law, it is based on the results of experiments, descriptions of which can be found in any standard book on Physics.

If V_0, ϱ_0 be the volume and density of the given quantity of gas at 0° C., and V, ϱ the corresponding quantities at t° C., and if a denotes its coefficient of expansion (which is equal to ·003665), then the law states that

$$V - V_0 = V_0 at,$$

or $$V = V_0(1 + at) \quad \ldots \quad (16)$$

Since mass $= V\varrho = V_0\varrho_0$, it follows that

$$\varrho_0 = \varrho(1 + at) \quad \ldots \ldots \quad (17)$$

103. If the temperature, t°, be negative, *i.e.* the temperature be lower than 0° C., this law is still true for a definite range of temperature provided the whole mass continues to be gaseous. Assuming the possibility of a contraction, in accordance with (16), to an indefinite extent, it gives that when $t = -\frac{1}{a}$ or -273, V is zero. The temperature -273° C. is called the *absolute zero* and the temperature measured from this point the *absolute temperature*. Thus if T° denotes the absolute temperature corresponding to t° C.,

$$T = t + \frac{1}{a} = t + 273. \quad \ldots \ldots \quad (18)$$

The utility of the absolute temperature is that it affords a shorter mode of representation of the formulae for the gases. For example, (16) and (17) can be written as

$$V = V_0 aT \quad \text{and} \quad \varrho_0 = \varrho aT. \quad \ldots \ldots \quad (19)$$

That is, *the volume of a given mass of gas varies directly, and its density varies inversely as its absolute temperature.*

The absolute zero, or $-273°$ C, has not been reached in practice, moreover, all known gases are liquefied at temperatures higher than $-273°$ C. We have therefore no experimental facts on which to base the actual behaviour of gases at absolute zero

104. By combining Charles' and Boyle's laws we get an important relation between the pressure, volume and temperature of a given mass of gas, viz $pV \propto (1+at)$, or $pV \propto T$.

To establish the relation, let p_0, p denote the pressures and V_0, V the volumes of the given mass of gas at 0° C. and $t°$ C respectively We can suppose that the change from p_0, V_0 to p, V takes place in two stages. Firstly, let the temperature be increased from 0° C to $t°$ C, keeping the pressure constantly at p_0, * let the volume change in consequence from V_0 to V' (say) Then by (16),

$$V' = V_0(1 + at)$$

Next change the pressure from p_0 to p so that the volume changes from V' to V, the temperature remaining constant. Therefore, by Boyle's law,

$$pV = p_0V' = p_0V_0(1 + at) = p_0V_0aT, \qquad \ldots \ (20)$$

if T denotes the absolute temperature As $\frac{1}{a} = T_0$, the absolute temperature corresponding to 0° C, we may write this as

$$\frac{pV}{T} = \frac{p_0V_0}{T_0}, \text{ which is constant} \quad \ldots\ldots(21)$$

105. Mixture of Gases. It has been seen experimentally that *if given volumes of two gases* (which are such that no chemical action takes place between them) *at the same*

* This can be managed with the help of the cylindrical vessel (with piston) described in Art 101

temperature and pressure are allowed to mix, the volume of the mixture, also at the same temperature and pressure, is equal to the sum of the given volumes of the gases

The above experimental fact enables us to establish the following rule .

If equal volumes, v, of two gases at the same temperature but at different pressures be mixed together and if the volume of the mixture (also at the same temperature) be v, then the pressure of the mixture is equal to the sum of the pressures of the two gases.

This rule can also be stated thus .

The pressure of the mixture of two gases occupying a given volume would be the sum of the pressures that each gas would have when occupying alone the same volume at the same temperature

To prove this rule of superposition of pressures, let us suppose that p_1, p_2 are the pressures of the two gases when the volume (of each) is v Let us change the pressure of the second gas to p_1 before mixing, so that its volume changes from v to v' (say) Then, by Boyle's law,

$$p_1 v' = p_2 v$$

When the gases are mixed, the volume of the mixture $= v + v'$ and its pressure p_1, by the experimental law stated in the beginning Finally, the volume of the mixture is changed to v; then the required pressure P will be given by

$$Pv = p_1(v + v') = p_1 v + p_2 v,$$

or $$P = p_1 + p_2. \qquad \text{(22)}$$

This rule can be applied to the mixture of any number of gases.

106. *If given masses of two gases whose volumes are v_1, v_2 and pressures p_1, p_2 respectively are mixed together, to find the pressure of the mixture when its volume is V, assuming the temperature to be the same throughout.*

Let us suppose that before mixing the volume of each gas is changed to V, then their pressures are, by Boyle's law,

$$\frac{p_1 v_1}{V} \text{ and } \frac{p_2 v_2}{V}$$

respectively. Now they are mixed; the pressure P of the mixture when its volume is V is, by (22), given by

$$P = \frac{p_1 v_1 + p_2 v_2}{V} \quad \ldots\ldots\ldots\ldots (23)$$

Alternative proof Let the pressure of the second gas be changed from p_2 to p_1, its new volume will then be $\frac{p_2 v_2}{p_1}$.

When the gases are mixed, the volume of the mixture is $v_1 + \frac{p_2 v_2}{p_1}$, and its pressure is p_1 by the experimental law, Art 105. Finally the volume of the mixture is changed to V, let its pressure be then P

$$\therefore\ PV = p_1\left(v_1 + \frac{p_2 v_2}{p_1}\right) = p_1 v_1 + p_2 v_2,$$

which is the same as (23)

This result can also be extended to the mixture of more than two gases

107. A barometer in which the vacuum (above mercury or the liquid in the tube) is not perfect is called faulty

Ex 1. A faulty barometer which contains some air above mercury, records 29 in and 27 5 in when true atmospheric pressures are 30 in and 28 in. of mercury respectively Find the lengths occupied by the air in the tube at the time of the readings and the volume of this quantity of air at atmospheric pressure of 30 in [Corrections for barometer including that due to capacity of the cistern not to be taken into consideration.]

Let a in be the length of the barometer tube above the mercury level in the cistern Then the length of the air

column in the tube will be $(a-29)$ in and $(a-27{\cdot}5)$ in when the atmospheric pressures are 30 in and 28 in respectively The pressure of this volume of air is equal to that at the mercury level in the tube, therefore the pressures in the two cases are $(30-29)$ in. and $(28-27{\cdot}5)$ in. of mercury, *i.e* equal to 1 in. and $\frac{1}{2}$ in of mercury, because the pressure at the cistern level is the atmospheric pressure

Let x in. be the length that would be occupied by the volume of air at 30 in pressure Then, by Boyle's law,

$$x \,.\, 30 = (a-29) \quad 1 = (a-27\ 5) \quad \tfrac{1}{2}$$

Solving we get $a = 30\ 5$ and $x = \ 05$

$\therefore$ the lengths of air column are 1 5 in, 3 in and ·05 in *Answer*

Ex 2 A cylinder contains equal volumes of two gases at 0° C, which are separated from each other by a thin weightless piston (of negligible volume) which accurately fits the cylinder and can move freely in it. If the temperature of one gas be raised to t° C, find the position of the piston assuming that the other gas remains at 0° C.

The piston divides the cylinder in the ratio of the volumes of the two gases To find the position of the piston is equivalent to determine this ratio

Let V be the volume and p the pressure of each gas in the beginning, *i e* $2V$ is the volume of the cylinder; let V_1 be the volume of the gas at t° C or $273° + t°$ absolute temperature, and V_2 that of the other gas at 0° C or 273° absolute temperature Since the piston is in equilibrium, the pressures of the two gases (which it separates) are equal, let p' denote their common pressure in the second case Then by (21),

$$\frac{p'V_1}{273+t} = \frac{pV}{273} \quad \text{and} \quad \frac{p'V_2}{273} = \frac{pV}{273}$$

$$\therefore \qquad V_1 \,.\, V_2 = 273 + t \quad 273,$$

i e the piston divides the axis of the cylinder in the above ratio

Ex 3 The volume, pressure and absolute temperature of a given mass of a gas are p_1, V_1 and $T_1°$, the corresponding quantities for another given quantity of a second gas are p_2, V_2 and $T_2°$. These are mixed together and the volume of the mixture brought to V and its temperature to $T°$ (absolute) If there be no chemical action between the gases, find the resulting pressure of the mixture

Let us suppose that the gases are brought to the temperature $T°$ before mixing ; the final result will not be affected by this supposition Let p_1', p_2' and V_1', V_2' be the pressures and the volumes of the two gases after this change in temperature Then, by (21),

$$\frac{p_1'V_1'}{T}=\frac{p_1V_1}{T_1} \text{ and } \frac{p_2'V_2'}{T}=\frac{p_2V_2}{T_2}.$$

Applying next the rule (23), we have the final pressure,

$$p=\frac{p_1'V_1'+p_2'V_2'}{V}$$

since the temperature remains at $T°$,

$$=\frac{T}{V}\left[\frac{p_1V_1}{T_1}+\frac{p_2V_2}{T_2}\right],$$

or

$$\frac{pV}{T}=\frac{p_1V_1}{T_1}+\frac{p_2V_2}{T_2} \qquad \text{.......(24)}$$

Ex 4 A hollow closed conical vessel, of height h, floats partially immersed in water with vertex downwards and axis vertical A hole is then made very near the vertex and water allowed to come into the vessel so that no air escapes from within If the vertex was originally at a depth b and H is the height of water barometer, prove that the new depth c (of the vertex) satisfies the relation

$$c^3-b^3=\left[c-\frac{H(c^3-b^3)}{h^3-(c^3-b^3)}\right]^3$$

Since the vertex was originally at depth b, the weight of the vessel

$$= \text{wt. of the water displaced}$$
$$= \tfrac{1}{3}\pi b^3 \tan^2\alpha \cdot w, \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

where α is the semi-vertical angle of the cone and w is the weight of unit volume of water. Let x be the depth of water inside the vessel in the final position, therefore its level is $(c - x)$ below the external surface of water and the volume of the water inside $= \tfrac{1}{3}\pi x^3 \tan^2\alpha$

$\therefore$ wt. of the cone + wt. of the water inside

= wt. of the water displaced,

or, from (i),

$$\tfrac{1}{3}\pi b^3 \tan^2\alpha \cdot w + \tfrac{1}{3}\pi x^3 \tan^2\alpha \cdot w = \tfrac{1}{3}\pi c^3 \tan^2\alpha \cdot w,$$

or
$$b^3 + x^3 = c^3 \quad \ldots\ldots \quad \text{(ii)}$$

Again, the air which occupied the whole volume of the cone at atmospheric pressure H, now occupies a volume $\tfrac{1}{3}\pi (h^3 - x^3) \tan^2\alpha$ under a pressure $H + c - x$, which is the pressure at the level of water inside the vessel. Therefore by Boyle's law,

$$\tfrac{1}{3}\pi h^3 \tan^2\alpha \cdot H = \tfrac{1}{3}\pi (h^3 - x^3) \tan^2\alpha \cdot (H + c - x),$$

or
$$h^3 H = (h^3 - x^3)(H + c - x) \quad \text{(iii)}$$

From (iii),
$$x = c - \frac{x^3 H}{h^3 - x^3};$$

$$\therefore x^3 = \left(c - \frac{x^3 H}{h^3 - x^3}\right)^3$$

Substituting the value of x^3 from (ii) we get the result.

EXAMPLES. 12.

1. Specific gravity of glycerine being 1.27 and that of mercury 13.6, find the height of a glycerine barometer when that of a mercury barometer is 30 in.

2. In the formula $p = k\rho$, calculate the numerical value of k in ft. sec. units from the following data: specific gravities of air and mercury are .0013 and 13.6 respectively, the height of the mercury barometer is 30 inches and the value of $g = 32.18$ ft. sec. units.

3. Bubbles of air rise from a depth of 20 ft. in water. Find the ratio in which the radius of a bubble is increased when it reaches the surface, given that the height of the mercury barometer is 30 in. and the sp gr. of mercury is 13·6.

4. A given mass of air at 23° C. and a pressure of 27 in. of mercury occupies a volume of 120 c c Find its volume at 60° C. and a pressure of 30 in of mercury.

5. A given mass of air is contained in a long cylindrical vessel fitted with an airtight piston which can move freely along the cylinder The temperature of the mass is changed without disturbing the surrounding atmosphere and no external force is applied to the piston Show that if the temperature changes through a series of values in A.P , the corresponding densities of the mass form an H P

6 A vertical tube closed at the top is allowed to sink vertically in deep water. The tube is of thin material, 60 cm long, 2 75 sq cm in cross-section and 80 grammes in weight. Find approximately the lengths of the parts into which the tube (when in equilibrium) is divided by the levels of water inside and outside the tube, the atmospheric pressure being due to a height of 10 metres of water

7. A conical glass, height h, is immersed mouth downwards in water ; find the depth of its rim when the water inside the glass rises half-way within it The height of the water barometer is H

8. At $t°$ C. the height of mercury and the length of Torricellian vacuum are h and a respectively A quantity of air which would occupy a length b of the barometer tube at 0° C. and a pressure due to height H of mercury, is introduced into the vacuum at the former temperature Show how the amount of fall of mercury in the barometer tube can be calculated

9. The constants k_1 and k_2 denote the values of 'k' in the formula $p = k\varrho$ for two gases Given masses m_1 and m_2 of the first and the second gas are mixed at the same temperature Prove that the value of the constant k for the mixture is

$$\frac{m_1 k_1 + m_2 k_2}{m_1 + m_2}.$$

10. The sp. gr of the gas inside a balloon is 0 1, the air at 76 cm. pressure and 0° C being taken as the standard substance. If the atmospheric pressure changes from 76 cm to 75 cm. while the temperature remains at 0° C., prove that the

lifting power of the balloon is reduced in the ratio 337 . 342, the volume of the balloon remaining unaltered and the weight of its envelope neglected

11. A barometer stands at 30 inches, the vacuum above the mercury being perfect; the area of the cross-section of the tube is 0 2 sq. in. If 0 2 cu in of ordinary air be introduced into the vacuum, the mercury is seen to fall through 3 in. Find the length of the original vacuum.

12. The readings of a true barometer and a barometer which contains a small quantity of air in the space above mercury are 30 and 28 inches respectively When both barometers are placed under the receiver of an air pump from which air has been partially exhausted, the readings are observed to be 15 and 14 5 inches Show that the length of the tube of the faulty barometer measured from the surface of mercury in the basin (which is wide) is 32 5 inches.

13. A faulty mercury barometer reads 28 in and 27 in. when a true barometer reads 28 5 in and 27 4 in respectively. Find the true barometric height when the reading by the faulty one is 26 in Also find the height of the faulty barometer when true atmospheric pressure is 28 in

14. A U-tube of uniform bore is fixed vertically with the bend uppermost and the open ends dipping into two wide basins containing mercury, a length a round the bend being occupied by air and the remainder by columns of mercury of length c One of the basins is then lowered through a small distance h Show that the mercury level in the corresponding tube is approximately lowered by $\dfrac{h(a+H-c)}{a+2H-2c}$, where H is the height of the mercury barometer, also find by how much the mercury rises in the other tube

15 If a small piece of glass (or metal) floats in mercury within the barometer tube without touching the sides, how would the height of mercury level in the tube be affected ?

16. A closed vessel in the shape of two coaxial cylinders, of equal height b, has one plane base completely covered up, the other base having only the area between the cylinders covered; the inner cylinder has a small hole very near its closed end. The vessel is placed with the closed end downwards and the axis vertical, and water is slowly poured into the inner cylinder till it is full If no air has escaped from the space between the cylinders, prove that its length is $\frac{1}{2}(\sqrt{h^2+4bh}-h)$, where h is the height of the water barometer.

17. A cylindrical tube of length a, open at both ends, is placed in a wide reservoir containing liquid (sp. gr. σ), with its axis vertical and one-third of its length immersed. A piston (which closely fits the tube) is then inserted from above and pressed till all the liquid in the tube is just expelled If h be the height of the water barometer, find the distance through which the piston has been moved in the tube.

18. In the last example, if the liquid be contained in a cylindrical cistern whose cross-section is n times that of the tube immersed, prove that the length of the air in the tube when all the liquid has been expelled would be

$$\frac{2ah(n-1)}{3(n-1)h+na\sigma}$$

Also show that this gives the same result as the previous example when n is very large.

19. A cylindrical vessel, of weight W and height a, just sinks in water when immersed with its open end downwards and a weight W placed on it at the top When it is placed in water with the closed end foremost it requires a downward force of $2W$ to sink it to the same depth as before Calculate the height of the water barometer

20. A heavy vessel, open at one end and made of thin sheet of metal, is of the shape of a surface of revolution and its capacity is V. It is immersed in water with the open end downwards, the plane of its rim being horizontal and the axis of symmetry vertical Find the condition so that the vessel may float, wholly or partially immersed, in equilibrium The density of air is to be taken negligibly small

21. If the vessel of the last example floats in equilibrium when just immersed, prove that the volume of the compressed air within the vessel will be equal to the volume of the water the vessel would displace if it were to float with the open end upwards

22. A cylindrical vessel of height 3 ft , open at the top, is placed with its axis vertical ; a solid right circular cone, of height 6 ft , is put on the top of the cylindrical vessel, which it fits like an airtight piston, the axis of the cone being vertical and vertex upwards The sp gr of mercury and the cone are 13·6 and 8 respectively, and the height of the mercury barometer is 30 in. If the cone be allowed to descend into the

cylinder, prove that the cone will come to a position of equilibrium when its base is at a distance of 2 04 ft. from the closed end of the cylinder

23. A piston of small thickness and of weight w per unit area fits closely a cylinder which contains given masses of two gases separated by the piston Initially the cylinder was kept with its axis horizontal, when it was observed that the piston was at the centre of the cylinder and the absolute temperatures of the two gases were $t_1°$ and $t_2°$ The cylinder was then turned, with the former gas uppermost, so that its axis became vertical and the gases were brought to the same temperature $t°$ Show that the pressures p_1 and p_2 of the gases satisfy the relations

$$p_2 - p_1 = w \quad \text{and} \quad \frac{1}{p_1 t_1} + \frac{1}{p_2 t_2} = \frac{2}{pt},$$

where p was the initial pressure of each gas

24. A straight tube of uniform narrow bore and closed at both ends, contains some mercury in the middle and air at each end When the tube is vertical the portions occupied by air are of lengths a and b respectively, which become α and β when the tube is inverted Find the lengths occupied by the air when the tube is horizontal

25. Into a glass cylindrical vessel half-full of water another glass cylindrical vessel of the same height but half the areal cross-section is gently lowered with the open end foremost until it rests on the bottom of the former cylinder ; the lengths of the air space and water column in the latter are observed Prove that the ratio of the height of the water barometer to the length of the air space is as the difference (d) of the levels of water outside and inside the smaller cylinder to the height of the water column in this cylinder.

Find, also, the work done upon the water, and express it in terms of d and the weight (W) of water per unit length of the larger cylinder.

26. A thin piston, without weight, fits into a vertical cylinder which is closed at its base and filled with atmospheric air Initially the piston is very near the top of the cylinder , if water be slowly poured on the top of the piston, show that the upper surface of the water will be lowest when the depth of the water (over the piston) is $\sqrt{ah} - h$, h being the height of the water barometer and a the height of the cylinder.

27. In a cistern barometer the long tube is suspended by a string which passes over a smooth pulley and has a definite counterpoise attached to the other end. Prove that the ratio

of the changes in the length of the barometer tube above the mercury level in the cistern to the corresponding changes in the height of the mercury column in the tube is equal to the ratio of the sectional area of mercury in the tube to the annular sectional area of the material of the tube.

108. If the density of air were the same at all heights, the height of the atmosphere above the surface of the earth could be easily calculated Thus let h be the height of the mercury barometer, ϱ the density of mercury, σ that of the homogeneous atmosphere (and equal in value to that of the air at the surface of the earth) and g the acceleration due to gravity (supposed to have the same value throughout this region). Then the pressure of air at the surface of the earth

= the weight of the column of air above earth (per unit cross-section)

$= g\sigma H,$

where H denotes the required height But this pressure is given by the barometer to be $g\varrho h$ Whence equating we get

$$g\sigma H = g\varrho h, \text{ or } H = \frac{\varrho}{\sigma} h \qquad \ldots (25)$$

Let $h = 30$ in.; $\varrho = 13{\cdot}6$, $\sigma = {\cdot}0013$, the density of water being taken as unity Substituting in the above, we get

$$H = \frac{13{\cdot}6 \times 5}{{\cdot}0013 \times 2} \text{ ft} = 4{\cdot}95 \text{ miles nearly}$$

The atmosphere being, however, a compressible fluid, its density varies from point to point, or from layer to layer, above a definite place on the earth's surface It depends, as we have seen, on the pressure and the temperature of the layer, the laws of variation of which are unknown The actual height of the atmosphere cannot therefore be definitely determined, but other considerations have led us to conclude that the atmosphere cannot extend indefinitely

beyond the earth's surface. Its height, however, far exceeds 5 miles, as has been calculated on the basis of homogeneity.

109. Determination of Altitudes by Barometer. If the atmosphere be assumed to be at rest under gravity (*i.e.* at rest relative to the earth), the only force acting on any element of air at vertical height z above the surface of the earth would be that of gravity. Therefore, by Arts. 72 and 73, we have

$$\frac{dp}{dz} = -g\rho \text{ or } dp = -g\rho\, dz, \quad . \quad . \quad . \quad . \quad (26)$$

since the direction of gravity is opposite to that of z increasing.* [Cf Art 78] On the right-hand side are involved g and ρ (*i e* the gravity and the density of air at height z) If the range of atmosphere under consideration be comparatively small we can take g and the temperature constant On this assumption we have, by Boyle's law,

$$p = k\rho.$$

$\therefore$ by division,
$$\frac{dp}{p} = -\frac{g}{k}\,dz,$$

or
$$\int_{p_1}^{p_2}\frac{dp}{p} = -\frac{g}{k}\int_{z_1}^{z_2} dz,$$

where p_1, p_2 denote pressures at heights z_1, z_2 respectively , let ρ_1, ρ_2 denote the densities at these altitudes Then

$$\log\frac{\rho_2}{\rho_1} = \log\frac{p_2}{p_1} = -\frac{g}{k}(z_2 - z_1), \quad \ldots\ldots \ldots\ldots (27)$$

or
$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} = e^{-\frac{g}{k}(z_2 - z_1)}. \quad \ldots\ldots \ldots\ldots (28)$$

The above formulae furnish us with a means of determining the difference of altitudes of two places by observing the atmospheric pressures there, provided the temperature

* This equation could be obtained from elementary principles, as in Art 72, by taking the z-axis vertically upwards and the elementary cylinder parallel to this axis.

be the same. The ratio $p_2 : p_1$ can be replaced by the ratio of the barometric heights, viz. $h_2 : h_1$.

Note. If the temperature at the two places be different, we may proceed on the assumption that the mean temperature of the atmosphere between the places is constant, *i e* $\frac{1}{2}(t_1+t_2)=t$ (a constant), so that we have to replace the constant k by its mean value $k_0(1+at)$ where k_0 is the value of k at 0° C. Thus (27) becomes

$$z_2 - z_1 = \frac{k_0}{g}(1+at)\log\frac{p_1}{p_2}$$

110. In the preceding article we had assumed the temperature to be constant, such a condition of air (or of any gas) is said to be *isothermal*

On the other hand, if we assume that the total amount of heat in a given mass of a gas is unchanged during any process, although the temperatures of its elements may alter, the condition (or the process) is called *adiabatic* Such a thing happens when a given mass of gas suddenly expands (or contracts), when a sound wave passes through a volume of gas, etc. During adiabatic processes the law connecting the pressure and the volume is

$$pv^{\gamma} = c', \qquad \ldots\ldots\ldots\ldots (29)$$

where γ and c' are constants Since $v=\frac{m}{\varrho}$ and m is unaltered so long as we confine our attention to a given mass, this may be written as

$$\varrho = cp^{1/\gamma}, \qquad \ldots\ldots\ldots\ldots (30)$$

where the constant c is equal to $m/(c')^{\frac{1}{\gamma}}$.

111. On the hypothesis that the state of the column of air is adiabatic, the equation

$$dp = -g\varrho\, dz$$

gives

$$p^{-\frac{1}{\gamma}}\, dp = -gc\, dz,$$

integrating, as in Art 109, we get

$$\frac{\gamma}{\gamma-1}\left[p_2^{1-\frac{1}{\gamma}}-p_1^{1-\frac{1}{\gamma}}\right]=gc(z_1-z_2) \quad \ldots \ldots \ldots (31)$$

112. *Ex* 1 Obtain the modification of the formula (27) [or of (28)] by taking into account the variation of gravity due to change of distance from the earth (*i e* from its centre, the earth being regarded as a sphere)

From (9) we have the value of gravity, g, at a distance z to be given by

$$g=\frac{g_0 a^2}{(a+z)^2},$$

where g_0 is the value of gravity at the earth's surface. Substituting in the equation (Art 109)

$$\frac{dp}{p}=-\frac{g}{k}dz,$$

we get

$$\frac{dp}{p}=-\frac{g_0 a^2}{k}\,\frac{dz}{(a+z)^2}.$$

$\therefore$ integrating as before,

$$\log\frac{p_2}{p_1}=\frac{g_0 a^2}{k}\left[\frac{1}{a+z_2}-\frac{1}{a+z_1}\right] \quad \ldots \ldots \ldots \ldots (32)$$

Ex 2 The volume of a given mass of gas at pressure p is v If the gas is compressed isothermally (Art 110), show that the work done in compressing it from volume v_1 to v_2 is $pv\log\frac{v_1}{v_2}$

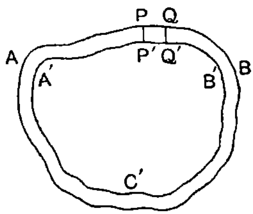


FIG 65

Suppose that during a small compression the volume of the gas is changed from v to $v+\delta v$ (δv being a negative quantity), and its envelope changed from Fig ABC to Fig $A'B'C'$, *i e* the volume between ABC and $A'B'C' = -\delta v$

Let PQ denote an element of area on the surface ABC, of magnitude α, let this element occupy the position $P'Q'$

after compression, so that PP' is normal to the surface ABC. The force acting on this element $= pa$, p being the pressure of the gas in volume ABC.

$\therefore$ the work done in bringing this element to $P'Q'$ $= -pa \cdot PP'$.

Adding up similar expressions for other elements, we get

the total work done in this small compression (from v to $v + \delta v$)

$$= -\Sigma pa \cdot PP' = -p \cdot \Sigma a \cdot PP'$$

$$= -p\, dv \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

because $a \cdot PP'$ is the volume of the element $PQQ'P'$, and $\Sigma a \cdot PP'$ is the sum of such elements, *i e* is the volume between ABC and $A'B'C'$.

$\therefore$ work done in compressing the gas from v_1 to v_2

$$= -\int_{v_1}^{v_2} p\, dv = -\int_{v_1}^{v_2} \frac{k_1}{v}\, dv,$$

since $pv = k_1$, by Boyle's law,

$$= k_1 \log \frac{v_1}{v_2} = pv \log \frac{v_1}{v_2} \quad \ldots \quad (34)$$

Note. In solving the above problem we had assumed that there was no pressure on the envelope ABC from outside, the work being done against the internal pressure p only. If, however, there be external pressure P (*e g.* atmospheric pressure), the work will be done against the resultant pressure $(p - P)$ The equations (33) and (34) would have to be modified in this case

EXAMPLES. 13.

1. Prove that the height of the homogeneous atmosphere (of density equal to that at the earth's surface) is equal to k/g, where k is given by the formula $p = k\rho$ and g is the acceleration due to gravity (assumed constant)

2. Determine roughly the height of the atmosphere if its density were throughout equal to one-half of the density of air at the earth's surface (viz. 0013), the effects of the variations of temperature and gravity being neglected

3. Prove that if p, p' are the pressures at two points the vertical distance between which is h, the ratio of the pressure to the density at any point is equal to $gh/\log\frac{p'}{p}$, the temperature being assumed to be constant

4. Assuming the atmosphere to be at constant temperature, prove that if points are taken at heights (above the ground) which are in A P , the corresponding pressures are in G P.

5. If p and p' are the pressures at two points whose vertical distance is h, find the pressure at an intermediate point which is at a vertical distance z above the lower point and express it in terms of z, h, p and p' The temperature may be taken to be uniform

6. A heavy gas at constant temperature is confined in a vertical cylinder of height h If ϱ be the density at the base, prove that the mean density $=\frac{k\varrho}{hg}\left(1-e^{-\frac{gh}{k}}\right)$.

7 If a change from 30 in to 27 in in the barometric height correspond to a rise in altitude of 2290 ft , find the rise in altitude which corresponds to the barometric height of 24 in. Given log 2 = 3010, log 3 = 4771

8 A balloon, whose height is *not* too large to permit of variation in the atmospheric pressure within this range, contains hydrogen and requires a force mg to prevent it from rising when its lowest point touches the ground Show that it can float with its lowest point at height h above the ground, where $h=\frac{k}{g}\log\frac{M+m}{M}$, M denoting the mass of the balloon with the enclosed gas

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9. A given mass of gas at pressure p occupies a volume v. Find the work done in compressing it adiabatically from volume v_1 to v_2

10. The pressure and density of air at the earth's surface are p_0 and ϱ_0 If the pressure of the air at any height varied as the mth power of its density, show that the height of the atmosphere would be equal to $\frac{m}{m-1}$ times the height of the

homogeneous atmosphere of density ϱ_0 and producing the pressure p_0 at the earth The variations of temperature and gravity are not to be taken into account.

11. Assuming that the temperature of the atmosphere in equilibrium diminishes upwards uniformly with a certain constant gradient, prove that its density will be uniform Also show that the gradient mentioned above is approximately 34·3° centigrades per kilometre Density of air at 0° C and 76 cm. pressure is 0013, density of mercury is 13 6 and g is to be taken as invariable

12. The fall in temperature in the atmosphere is proportional to the increase in height above the earth's surface ; h, h' are the readings of mercury barometer at two stations, the former of which is at a height z above the other. Prove that

$$z = \lambda\left[1 - \left(\frac{h}{h'}\right)^m\right],$$

where λ and m are some constants.

13. A closed tube AB containing air is made to rotate uniformly in a horizontal plane about the end A. Prove that, when the air is in relative equilibrium, the density at B exceeds that at A in the ratio $e^{v^2/2gH}$ 1, where v is the velocity of the end B and H is the height of the homogeneous atmosphere.

14. A piston of weight w rests in a vertical cylinder of cross-section a, being supported by a column of air below of length h If the piston receives a vertical blow of impulse P, which forces the piston down through a distance x before it comes to rest, prove that

$$(w + \Pi a)\left(x + h\log\frac{h-x}{h}\right) + \frac{gP^2}{2w} = 0,$$

where Π is the atmospheric pressure and the compression is isothermal.

CHAPTER X

DETERMINATION OF SPECIFIC GRAVITY. INSTRUMENTS

113. We have seen in Art 2 that the specific gravity of a substance can be defined by either the formula

$$\frac{\text{mass of the substance}}{\text{mass of an equal volume of water}}, \quad \ldots\ldots\ldots (1)$$

or the formula

$$\frac{\text{volume of equal mass of water}}{\text{volume of the substance}} \quad \ldots\ldots (2)$$

The methods to be described in this chapter are based on (1) or on (2), *i e* they help to determine the mass of equal volume or the volume of equal mass, of water In practice the effect of the atmosphere is neglected, for, as has been shown in Art 43, it is generally very small In some examples, however, it has been shown how this effect can be determined and the correct value of the specific gravity obtained

The sp gr of solids or liquids are usually determined with the help of the following instruments ·

(1) The hydrostatic balance,

(2) The specific gravity bottle,

(3) The U-tube, and

(4) Hydrometers.

We shall now consider them in order

THE HYDROSTATIC BALANCE

114. This is an ordinary balance, with the difference that one of the pans is suspended by shorter chains (or wires) than the other, so that the level of the former is much higher than that of the latter, on the under-side of the higher pan is a hook. The object of this arrangement is that weighments can be made while a solid body is hung from the hook in suspension in air or in some liquid contained in a vessel placed underneath this pan *

115. *To determine the specific gravity of a liquid.*

Take a solid body which neither melts in, nor produces chemical changes with, nor floats in the given liquid as well as water Find its weights while the solid is suspended in air, in water and in the liquid respectively, let them be W_1, W_2 and W_3. Therefore, neglecting atmospheric effects,

weight of the body $= W_1$,

wt. of the body − wt. of the water displaced $= W_2$,

and wt of the body − wt of the liquid displaced $= W_3$

[Cf Ex 2, Art 54.]

∴ weights of equal volumes of water and liquid

$$= W_1 - W_2 \text{ and } W_1 - W_3$$

respectively.

∴ by formula (1), the sp gr of the liquid

$$= \frac{W_1 - W_3}{W_1 - W_2} \quad \ldots\ldots\ldots\ldots \quad (3)$$

116. *To determine the specific gravity of a solid heavier than water*

(1) If the solid does not melt in water (nor produces any chemical changes with it), find its weights while in suspen-

* An ordinary balance, with a wooden bridge of sufficient breadth to allow free movement of the pan across which it is placed, may serve the purpose, the body being suspended from the hook (in the arm of the balance) which supports the pan, the vessel of liquid (for immersion of the body) can be placed on the bridge

sion in air and in water, let them be W_1 and W_2. As in the last example,

the weight of the body $= W_1$

and the weight of the water displaced $= W_1 - W_2$,

the latter having the same volume as the solid.

$\therefore$ by (1), the sp. gr. of the solid

$$= \frac{W_1}{W_1 - W_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

(ii) If, however, the solid does not conform to the conditions stated in (i), take another liquid in which the solid does not melt nor does it react chemically with the former. The sp. gr. of this liquid is determined (Art. 115); let it be σ. Find the weights of the solid while suspended in air and in the liquid separately; let them be W_1 and W_2. Then

the wt. of the solid $= W_1$,

and the wt. of the liquid displaced $= W_1 - W_2$.

$\therefore$ the wt. of an equal volume of water $= \dfrac{W_1 - W_2}{\sigma}$

$\therefore$ by (1), the sp. gr. of the solid

$$= \frac{W_1 \sigma}{W_1 - W_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

117. *To find the specific gravity of a solid lighter than water.*

We shall assume that the solid has no reaction with water as stated in (i) of the previous article; otherwise we have to proceed in a manner similar to the method indicated in (ii) of the same article.

Take another solid (called the sinker) heavier than water and sufficiently heavy to cause the given (light) body to sink along with it when tied together and put in water.

Find the weight of the given solid; next find the weight of the solid and the sinker when both are suspended in water, and finally the weight of the sinker alone when

hanging in water, let them be W_1, W_2 and W_3 respectively Then

W_2 = wt. of the solid and the sinker in water
= wt. of the solid + wt. of the sinker − wt of water displaced by the solid − wt. of water displaced by the sinker,

and W_3 = wt. of the sinker in water
= wt of the sinker − wt of water displaced by the sinker.

∴ $W_2 - W_3$ = wt of the solid − wt of water displaced by the solid.

∴ wt of water displaced by the solid (which is of equal volume) $= W_1 - (W_2 - W_3)$

∴ by (1), the sp gr required

$$= \frac{W_1}{W_1 - W_2 + W_3} \quad \ldots \quad (6)$$

118. *Ex* 1 A solid weighs w in air and w_1, w_2 in two liquids respectively Compare the specific gravities of the liquids

It follows from (1) that, if σ_1, σ_2 be the sp gr of the two liquids, the ratio $\sigma_1 : \sigma_2$ equals the ratio of the masses (or weights) of equal volumes of these liquids

Now, wt of the first liquid displaced by the solid $= w - w_1$, and wt. of the second liquid displaced by the solid $= w - w_2$ Obviously their volumes are equal

$$\therefore \sigma_1 : \sigma_2 = w - w_1 : w - w_2$$

Ex 2 The sp gr of a solid heavier than water is found to be μ by the hydrostatic balance when the effect of air is neglected Prove that if this effect be taken into consideration, the true sp gr is $\mu(1-\sigma)+\sigma$ where σ is the sp gr. of the atmospheric air

In this method there are two weighings (Art 116), and the equations given in that article are approximate in-

asmuch as the effect of air is neglected We shall now write down the exact equations in each case

It has been shown in Note, Ex. 2, Art. 54, that in forming the equations the consideration of the two pans may be omitted ; it will be sufficient for our purpose, therefore, to consider the solid and the weight-pieces only Let the sp gr of the latter be ϱ, the volume and the true sp gr. of the solid be V and μ_0 respectively, and w denote the weight of unit volume of water

The downward pull on the arm of the balance carrying the solid in the first weighment

$$= \text{wt of the solid} - \text{wt of the air displaced by it}$$
$$= (V\mu_0 - V\sigma)\, w\,,$$

and the downward pull on the other arm

$$= \text{wt of the weight-pieces} - \text{wt of the air displaced by them}$$
$$= W_1\left(1 - \frac{\sigma}{\varrho}\right),$$

since W_1 denotes the true weight of the pieces, their volume is $\frac{W_1}{\varrho w}$, and so the weight of an equal volume of air is $\frac{W_1\sigma}{\varrho}$ Hence, equating, we get

$$Vw(\mu_0 - \sigma) = W_1\left(1 - \frac{\sigma}{\varrho}\right) \qquad \ldots.(\text{i})$$

Similarly for the second weighment, we shall get

$$Vw(\mu_0 - 1) = W_2\left(1 - \frac{\sigma}{\varrho}\right), \qquad \ldots \ldots (\text{ii})$$

the body being immersed in water Therefore, by division,

$$\frac{\mu_0 - \sigma}{\mu_0 - 1} = \frac{W_1}{W_2},$$

whence $\frac{\mu_0 - \sigma}{1 - \sigma} = \frac{W_1}{W_1 - W_2} = \mu$, from (4), Art 116

$$\therefore\ \mu_0 = \mu(1 - \sigma) + \sigma$$

EXAMPLES. 14.

1. A given body weighs three times as much in air as in water, prove that its sp gr. is 1·5.

2. An alloy made of gold (sp gr 19 25) and silver (sp gr. 10 5) weighs 11 7 oz in air and 10 8 oz in water Find the volumes of gold and silver in the alloy, it being given that 1 cu. ft of water weighs 1000 oz.

3. If the weight of the sinker (sp. gr 9) be $4\frac{1}{2}$ times that of the solid whose sp gr is to be determined, and if the apparent weight (in water) of the sinker and the solid together be $2\frac{1}{2}$ times the weight of the solid, find its sp. gr

4. A piece of brass (sp gr 8 4) and a piece of iron (sp gr. 7 8) are suspended from the two arms of a balance, the former being immersed in water and the latter in alcohol (sp gr 0 8) If the balance be in equilibrium with its arms horizontal, find the ratio of the volumes of the pieces, as also the ratio of their masses.

5. A piece of wood placed on one pan of a balance is balanced by weights of denomination w placed on the other pan, is the true weight of the wood greater or less than w? Give reasons

6 The value of the sp gr of a liquid is found, as in Art 115, to be ϱ (>1). Prove that this value is really too large by the amount $\sigma(\varrho - 1)$, where σ is the sp gr of air

7. A ring consists of gold, diamonds and rubies It weighs (*in vacuo*) 44 5 grains and in water 38 75 gr. When all the rubies are taken out, the ring weighs 34 75 gr in water Find the weight of the gold, diamonds and the rubies in the ring separately. Given that the sp gr of gold, diamond and ruby are 19 25, 3 5 and 3 respectively

SPECIFIC GRAVITY BOTTLE

119. The specific gravity bottle is one which can hold a definite volume of liquid ,' for this purpose it is fitted with a glass stopper which is perforated along its axis so that, when the stopper is pushed in the bottle (filled with liquid), the excess of the liquid passes out through this hole and the stopper is always able to occupy the same position (when pushed home)

(*a*) *To determine the sp gr. of a liquid*

Weigh the bottle when empty , next weigh it when filled with water, and again when filled with the liquid Let these weights be W_1, W_2, W_3 respectively

Then the weight of water in the bottle $= W_2 - W_1$,

and ,, liquid ,, $= W_3 - W_1$.

$\therefore$ by (1), the sp. gr of the liquid

$$= \frac{W_3 - W_1}{W_2 - W_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

(*b*) *To determine the sp. gr of a solid heavier than water.*

Let the solid be as stipulated in Art 115 , break it into small pieces so that they can be put inside the bottle Weigh the bottle (1) when empty, (2) when solids are put in it, (3) when solids are put in it and then filled with water, and (4) when filled only with water , let them be W_1, W_2, W_3, W_4* Then

wt of the solid $= W_2 - W_1$,

wt of the whole vol of water $= W_4 - W_1$,

and wt of the solid + wt of the remaining vol. of water

$$= W_3 - W_1$$

$\therefore$ subtracting the second from the third,

wt of the solid − wt of an equal vol of water

$$= W_3 - W_4$$

$\therefore$ wt of the equal vol of water $= (W_2 - W_1) - (W_3 - W_4)$

$\therefore$ by (1) the sp gr of the solid

$$= \frac{W_2 - W_1}{(W_2 - W_1) - W_3 + W_4}. \quad \ldots\ldots\ldots\ldots(8)$$

(*c*) If the solid be soluble in water we take another liquid which has no effect on it Find the sp gr of the

* If convenient, the solid pieces can be weighed on the pan of the balance ; then it does not become necessary to take the first two weights The sp gr would be given by the formula (8) if $(W_2 - W_1)$ is replaced by the weight, W, of the solid pieces

liquid [cf (*a*)], let its value be σ Then proceed as in (*b*), using the liquid instead of water, and we shall obtain

wt of the equal vol of the liquid

$$= W_2 - W_1 - W_3 + W_4$$

$\therefore$ wt. of an equal vol of water

$$= (W_2 - W_1 - W_3 + W_4)/\sigma$$

$\therefore$ the sp gr of the solid

$$= \frac{(W_2 - W_1)\sigma}{(W_2 - W_1) - W_3 + W_4} \qquad \ldots\ldots\ldots\ldots (9)$$

(*d*) If the solid be lighter than water, we take, if possible, another liquid into which the solid sinks, we then proceed as in (*c*)

THE U-TUBE

120. This apparatus is used for the determination of sp gr of liquids only

(*a*) If the liquid and water do not mix, they are poured into the two vertical arms of the U-tube, one down each arm Let B be their surface of separation, AB, BDC being water and the liquid Take two points, one in each arm, *in the same liquid* and in the same horizontal level Let E be at the same level as B, then B and E are suitable points (or levels) Measure the heights AB and EC, let them be h_1 and h_2

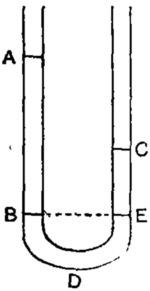

FIG 66

Then pressure at B = pressure at E (Art 13),

or $$\Pi + wh_1 = \Pi + w\sigma h_2 \text{ (Art 11)},$$

where Π is the atmospheric pressure, σ is the sp gr. of the liquid and w the weight of unit volume of water.

$$\therefore \sigma = h_1/h_2 \qquad \ldots\ldots\ldots\ldots (10)$$

(*b*) If the liquid mixes with water, some mercury is first poured into the U-tube, let BDE denote the portion occupied by mercury, BE being horizontal Next sufficient quantities of water and the liquid are poured down the branches AB and CE respectively, so that the levels of the mercury are the same as before Measure the columns, AB of water and EC of the liquid The calculation is the same as before, the required sp gr. being $AB : CE$

121. *The inverted* U*-tube* This apparatus contains an open tube fixed in the U-tube at the bend K (Fig 67) The lower (open) ends of the U-tube dip into vessels, E and F, containing water and the liquid whose sp. gr is to be determined The projecting tube at K is connected to an exhaust pump so that the pressure of the air in the portion BKD becomes less than the atmospheric pressure. In consequence, the liquids rise in the two branches up to B and D, measure the heights of the water column AB and the column CD of the liquid

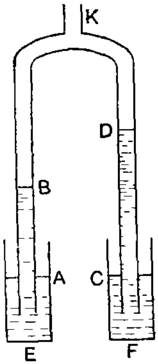

FIG 67

Since pressure of the air in the portion BKD

$$= \text{pressure at the level } B,$$

also $$= \text{pressure at the level } D,$$

$$\therefore \Pi - w \,.\, AB = \Pi - w\sigma \,.\, CD, \text{ as in Art } 120,$$

whence $$\sigma = AB : CD$$

This apparatus is also called *Hare's hydrometer*

EXAMPLES 15.

1. A sp. gr. bottle weighs 63 grammes when it is filled with water; some pieces of metal (sp gr 8 4) are put into it when some water overflows The bottle is then wiped on the outside and found to weigh 100 grammes Find the weight of the water that has overflowed

2. A sp. gr. bottle can contain 50 grammes of water. When a piece of silver (sp. gr. 10 5) is put in and the bottle filled up with water the contents weigh 97 5 grammes. Again, when the same piece of silver is put in and the bottle filled with a mixture of equal parts (by volume) of alcohol and water, the contents weigh 93 grammes. Find the sp. gr. of alcohol.

3. Taking into consideration the effect of the atmosphere, find the true sp. gr. of a liquid in terms of the value, ρ, found in Art. 119 (a) and the sp. gr , σ, of the atmospheric air.

4. The lower part of a U-tube contains mercury , a liquid is poured down one branch till it occupies a length a of the tube When water is poured into the other branch up to a height b above mercury, it is observed that the mercury level in this branch is lower than that in the other by c. If σ denote the sp. gr. of mercury, find the sp gr. of the liquid.

5. If in the previous example a thin cylinder of wood be put into the arm containing water, and if the height of the water column be increased by x when the cylinder floats in equilibrium (in contact with water only), prove that the mercury level in the other arm is raised by $\frac{x}{2\sigma}$

COMMON HYDROMETER

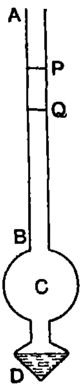

FIG. 68.

122. The specific gravities of liquids only can be determined by the common hydrometer It is generally made of glass, and consists of a long stem AB of uniform cross-section terminating in a bulb C, below which there is another smaller bulb D , the lower bulb is loaded with mercury (or lead shots) so that the instrument may float in a liquid with the stem in a vertical position The stem AB is graduated in such a manner that the graduation at the point P up to which the hydrometer sinks in a liquid gives the sp. gr of the liquid.

It is clear that the hydrometer always displaces liquids whose weights are equal to that of the instrument ; for this reason it is sometimes called the constant weight hydrometer.

123. *To find the specific gravity of a liquid*

Let V denote the volume of the hydrometer, supposed to be known. Put it into water, and let the portion AP (Fig. 68) remain above water when the instrument floats in equilibrium Next, put it in the liquid and let the length AQ remain above the surface Let $AP = a$, $AQ = x$ and the cross-section of the stem $= \alpha$, then

$$\text{the volume of water displaced} \qquad = V - a\alpha,$$

and $$\text{the volume of the liquid displaced} = V - x\alpha$$

These volumes have equal weights, *i e* equal masses

$\therefore$ by (2), the sp gr of the liquid

$$= \frac{V - a\alpha}{V - x\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

124. *To graduate a common hydrometer*

By putting the instrument in water and various other liquids whose sp gr are known (by other methods), the points up to which it sinks are marked and numbers equal to the respective specific gravities are put against the marks It is rather impracticable to use the same hydrometer for measuring the sp gr of all liquids, for such a hydrometer must have its stem of an inconvenient length In practice, therefore, one instrument is used for a definite range of specific gravities, another for a consecutive higher (or lower) range, and so on

Theoretically we proceed thus :

(i) Let s and s' be the sp gr corresponding to the values x and x' of AQ (the portion of the stem that remains above the liquid) Then, from (11),

$$s = \frac{V - a\alpha}{V - x\alpha} \text{ and } s' = \frac{V - a\alpha}{V - x'\alpha},$$

or $$x = \frac{V}{\alpha} + \left(a - \frac{V}{\alpha}\right) \cdot \frac{1}{s}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and a similar expression for x'.

If V, a and α are known (by measurements), the value of x corresponding to the sp gr s can be calculated from (12) The number s is then put against a mark made at Q, where $AQ = x$ Similarly for other graduations In this manner the whole stem can be graduated

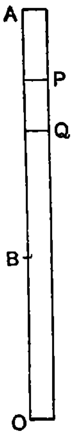

FIG 69

(ii) Instead of graduating from the top as above, the graduations may be made from below, as is generally done in practice.

Assume the stem prolonged to O so that the volume of the tube from O to A = the volume of the instrument That is, $OA \quad \alpha = V$

$$\therefore \text{ (11) gives the sp gr } = \frac{OA \quad \alpha - AP \,.\, \alpha}{OA \,.\, \alpha - AQ \quad \alpha},$$

or

$$s = \frac{OP}{OQ} \quad \ldots \ldots \ldots \ldots \quad (13)$$

As O would be a definite point on the prolongation of the stem, we may take O as our origin , and since P denotes the point up to which the hydrometer sinks in water, OP may be taken to be of definite length

$\therefore$ we see that, *if we vary OQ through a series of values in A P , the corresponding value of s (or sp gr) would be in H P ; conversely, if s varies through a series of values in A P , the corresponding values of OQ (or the distances of the points of graduation) would be in H P*

125. If we subtract from (12), the corresponding equation in x' and s', we obtain

$$x - x' = \lambda\left(\frac{1}{s} - \frac{1}{s'}\right), \ldots \ldots \ldots \ldots \quad (14)$$

where λ is put for $a - \dfrac{V}{\alpha}$, which is constant Similarly we shall have

$$x - x'' = \lambda\left(\frac{1}{s} - \frac{1}{s''}\right),$$

and so on. This shows that

if $\qquad x, x', x'', \ldots$ are in A P ,

$$\frac{1}{s}, \frac{1}{s'}, \frac{1}{s''}, \quad \text{are also in A P ,}$$

$$\therefore \ s, s', s'', \ldots \ \text{are in H P}$$

That is, if the stem is graduated at equal distances, the corresponding values for the sp gr must be in H P.

NICHOLSON'S HYDROMETER

126. This hydrometer can be used to ascertain the sp gr. of liquids as well as solids It is generally made of brass or any other metal It consists of a hollow cylindrical vessel C at one end of which is fixed a thin stem B. A pan, A, is attached to this stem at right angles to it From the other (*i e* the lower) end of the vessel C is suspended a cup or basket D, which is sufficiently heavy to make the instrument float in stable equilibrium with the stem B vertical There is a well-defined mark on the upper stem, and in using the hydrometer weight-pieces are placed on the pan till the instrument sinks in the liquid just up to this mark It is thus clear that this hydrometer is made to displace the same volume of the liquids (equal to the volume of the instrument from the mark on B downwards). For this reason it is also named as the *constant volume hydrometer*

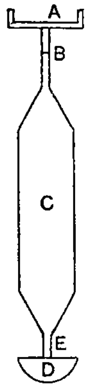

FIG 70

127. *To determine the sp gr of a liquid*

Weigh the hydrometer , let its weight be W. Put the hydrometer in water, and place sufficient weight-pieces on the pan A to sink the instrument up to the mark on B , let the weights put on the pan be W_1 Repeat the experiment with the liquid (whose sp gr is to be determined)

instead of water, and let W_2 be the weights on the pan Then

$$W + W_1 = \text{weight of the instrument and the weight-pieces}$$
$$= \text{weight of the water displaced ,}$$

also

$$W + W_2 = \text{wt of the liquid displaced}$$

Since their volumes are equal, the sp gr of the liquid, by (1),

$$= \frac{W + W_2}{W + W_1} \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots(15)$$

128. *To determine the sp gr of a solid heavier than water*

Put the hydrometer in water, and place weight-pieces till it sinks up to the mark , let the weight on the pan be W_1 Put the solid on the pan, and remove some of the weight-pieces till the hydrometer floats as before ; let the remaining weights (on the pan) amount to W_2 Next, put the solid on the lower pan, and adjust the weights on the upper pan to make the hydrometer sink to the mark , let the weights on the pan be W_3 Then

wt of the water displaced by the instrument

$$= \text{wt of the hydrometer} + W_1,$$

also $= \text{wt of the hydrometer} + \text{wt. of the solid} + W_2.$

$$\therefore \text{ wt of the solid} = W_1 - W_2$$

From the last observation we have, next,

wt of water displaced by the hydrometer + wt of water displaced by the solid

$$= \text{wt. of the hydrometer} + \text{wt of the solid} + W_3$$

$\therefore$ wt of water displaced by the solid

$$= W_3 - W_2$$

$\therefore$ by (1), the sp gr of the solid

$$= \frac{W_1 - W_2}{W_3 - W_2} \quad \ldots \quad \ldots \quad \ldots(16)$$

If the solid be lighter than water, the help of a sinker is required as in Art 117.

129. *Ex* 1 Show how to find the true sp gr. of a liquid by a Nicholson's hydrometer if the effect of air be taken into consideration [See Art. 127]

From (15) we have the approximate sp gr ,

$$\mu = (W + W_2)/(W + W_1),$$

where the W's denote the true weights of the weight-pieces (being the denominations marked on them)

Let μ_0 be the true sp gr of the liquid, σ and d the sp gr of the air and the weight-pieces and w the weight of unit volume of water Also, let V_1 be the volume of the hydrometer below the mark and V_2 that of the portion above it There are three observations made in Art 127, and we have to write the exact equations for them instead of the approximate ones given there For the first observation the correct equation is

wt of the hydrometer − wt. of air displaced by it

= wt of the pieces − wt of air displaced by them,

the effects of the two pans of the balance cancelling each other

Or, wt of the hydrometer $-(V_1 + V_2)\sigma w = W\left(1 - \frac{\sigma}{d}\right)$ (i)

For the second observation the exact equation is

wt of the hydrometer + weight of the pieces

= wt of the water displaced + wt of the air displaced by the upper portion of the hydrometer and the pieces,

or wt of the hydrometer $+ W_1 = V_1 w + V_2 \sigma w + W_1 \frac{\sigma}{d}$

$\therefore$ by (i), $(W + W_1)\left(1 - \frac{\sigma}{d}\right) = V_1 w(1 - \sigma)$ . . ..(ii)

Similarly for the third observation we have

wt. of the hydrometer $+ W_2 = V_1\mu_0 w + V_2\sigma w + W_2 \frac{\sigma}{d}$,

which with the help of (i) reduces to

$$(W + W_2)\left(1 - \frac{\sigma}{d}\right) = V_1 w (\mu_0 - \sigma) \quad \dots\dots\dots\text{(iii)}$$

Dividing (iii) by (ii) we get

$$\mu = \frac{W + W_2}{W + W_1} = \frac{\mu_0 - \sigma}{1 - \sigma},$$

whence $$\mu_0 = \mu(1 - \sigma) + \sigma$$

Ex 2 A common hydrometer whose volume is V and the cross-section of whose stem is $\frac{V}{c}$, has lengths a and b of its stem uncovered when floating in two liquids separately. In a mixture of the two liquids in the proportion m 1 by weight, a length x of the stem remains above the surface, while in a mixture in the same proportion by volume, y remains uncovered Prove that

$$c(x - y) + y(a + b) = xy + ab.$$

Let $p =$ the uncovered length of the stem when the hydrometer is put in water, let s_1, s_2 be the sp gr. of the given liquids and σ_1, σ_2 of the two mixtures Then from (11) we have

$$s_1 = \frac{c - p}{c - a}, \quad s_2 = \frac{c - p}{c - b}, \quad \sigma_1 = \frac{c - p}{c - x}, \quad \sigma_2 = \frac{c - p}{c - y} \quad \dots\dots\text{(i)}$$

Next, to find σ_1 in terms of s_1 and s_2, let M be the mass of the second liquid in the mixture, therefore mM is that of the first liquid, so that the mass of the mixture $= M(m + 1)$ Also the volume of the second liquid in the mixture $= \frac{M}{s_2}$ and that of the first $= \frac{mM}{s_1}$, so that the volume of the mixture $= M\left(\frac{m}{s_1} + \frac{1}{s_2}\right)$

$\therefore$ the sp gr. of the mixture $=\dfrac{\text{mass}}{\text{volume}}$,

or $$\sigma_1=\frac{m+1}{\dfrac{m}{s_1}+\dfrac{1}{s_2}}, \quad \text{or} \quad \frac{\sigma_1}{s_1}=\frac{m+1}{m+\dfrac{s_1}{s_2}}$$

Substituting from (i),

$$\frac{c-a}{c-x}=\frac{m+1}{m+\dfrac{c-b}{c-a}},$$

whence $$m(a-x)=x-b \qquad \ldots \ldots \ldots \quad \text{(ii)}$$

Again, in the second mixture, let V be the volume of the second liquid and mV that of the first, therefore their masses are Vs_2 and mVs_1 respectively The volume and the mass of the mixture are therefore

$$V(m+1) \quad \text{and} \quad V(ms_1+s_2)$$

respectively, so that the sp gr of this mixture is given by

$$\sigma_2=\frac{ms_1+s_2}{m+1}, \quad \text{or} \quad \frac{\sigma_2}{s_1}=\frac{m+\dfrac{s_2}{s_1}}{m+1}$$

Substituting from (i),

$$\frac{c-a}{c-y}=\frac{m+\dfrac{c-a}{c-b}}{m+1},$$

whence $$m(a-y)=\frac{c-a}{c-b}(y-b) \qquad \ldots\ldots\ldots\ldots \text{(iii)}$$

Eliminating m from (ii) and (iii), and removing the factor $(a-b)$ from the result, we shall have

$$cx+y(a+b-c)-(xy+ab)=0,$$

whence the required answer readily follows

Ex. 3 A common hydrometer is graduated to give correct readings at temperature $t°$. When it is placed in a

liquid which is at $t_1°(>t°)$, the hydrometer being at $t°$, the sp. gr. appears at first to be σ, but afterwards to be σ_1. Show that the true sp. gr of the liquid at $t°$ is

$$\sigma + \frac{\beta'}{\beta}(\sigma_1 - \sigma),$$

where β and β' are the coefficients of cubical expansion of the hydrometer and the liquid respectively.

At the first reading the hydrometer was at $t°$, there being no time for it to take up heat from the liquid Therefore the true sp gr of the liquid (which is at $t_1°$) $= \sigma$

Since the volume of the liquid expands by β' for every unit volume per degree (of temperature), its volume increases in the ratio $1 + \beta'(t_1 - t)$ 1 as its temperature changes from $t°$ to $t_1°$, and its mass is unchanged,

$$\therefore \text{ its sp gr at } t° = \sigma[1 + \beta'(t_1 - t)]. \quad \dots . \qquad \text{(i)}$$

When the hydrometer is heated from $t°$ to $t_1°$ (in contact with the liquid) its volume increases in the ratio

$$1 + \beta(t_1 - t) \quad 1,$$

and it gives, therefore, a reading corresponding to σ_1

Since the volume of the hydrometer increases and the density of the liquid remains the same (the temperature of the latter remaining at $t_1°$), the volume of the liquid displaced (whose weight is equal to that of the hydrometer) remains unchanged

$$\therefore \; V - xa = (V - x_1 a)[1 + \beta(t_1 - t)], \dots\dots \qquad \text{(ii)}$$

where V denotes, at $t°$, the volume of the hydrometer, a the cross-section of the stem, and x, x_1 the distances from the top, of the graduations marked σ, σ_1 respectively. It is to be noted that the volume of the hydrometer up to the mark σ_1 was $(V - x_1 a)$ at $t°$, and this finally increased to $(V - x_1 a)[1 + \beta(t_1 - t)]$.

But since σ, σ_1 correspond to x, x_1, we have from (ii)

$$\frac{\sigma_1}{\sigma}=\frac{V-xa}{V-x_1a}=1+\beta(t_1-t), \text{ from (ii)}$$

$$\therefore\ \sigma(t_1-t)=\frac{\sigma_1-\sigma}{\beta}$$

Substituting in (i) we get the desired result.

EXAMPLES. 16.

1. The volume of a common hydrometer is 15 c c and its weight is 10 grammes, the cross-section of its stem is 0 25 sq. cm in area. How much of it will be uncovered when the hydrometer is put in a liquid of sp. gr 0 8?

2. A common hydrometer floats in water with 4 cm. of its stem unimmersed, and in a liquid of sp gr 0 84 with 0 5 cm. of its length above the liquid Find the sp gr. of the liquid in which the hydrometer floats with 2 cm uncovered.

3. A Nicholson's hydrometer weighs 27 grammes; when weight-pieces of aggregate weight 23 grammes are placed on the upper pan, the hydrometer sinks up to the mark in water. How much weight should be placed on the pan so as to sink it up to the mark in a liquid of sp. gr 1 78?

4. A Nicholson's hydrometer 12 oz in weight requires 3 oz. on the pan to sink it up to the mark in pure alcohol (sp gr 0 795). When it is put into a mixture of alcohol and water it is found to require 5 oz to sink to the mark. Find the ratio (by volume) of alcohol and water in the mixture.

5. A common hydrometer sinks to the points A, B, C in liquids whose densities are ϱ_1, ϱ_2, ϱ_3 respectively. If $AB=a$, $BC=b$ and $AC=a+b$, prove that

$$\frac{b}{\varrho_1}+\frac{a}{\varrho_3}=\frac{a+b}{\varrho_2}$$

6. If a common hydrometer just floats in water, find the graduations that will give specific gravities increasing in a G P with common ratio r

7. Describe the observations to be made with a Nicholson's hydrometer in order to determine the sp gr. of a solid lighter than water, and calculate its value from the observations made, neglecting the effect of the atmosphere

8. If a small length of the stem (at its upper end) of a graduated common hydrometer be removed, show that its readings must all be reduced in a constant ratio so that it may still be used for measuring specific gravities

9. A common hydrometer floats in water with half the stem immersed A lighter liquid which does not mix with water is poured into the containing vessel until one-fourth of the stem remains above the liquid If the depth of the upper liquid be $\frac{1}{n}$th of the stem, n being < 4, find the sp gr of the liquid.

10. If the effect of the atmosphere be taken into consideration, show that the sp. gr. of a solid (heavier than water) will be $\mu(1-\sigma)+\sigma$, where μ is the value given by the formula (16) and σ is the sp. gr of the atmospheric air.

11. It is found that a solid whose true weight is w, placed on the upper pan, sinks a Nicholson's hydrometer to the mark, as also another solid whose true weight is W, when placed in the lower cup If the two solids have the same sp gr show that its value is $\frac{W-w\sigma}{W-w}$, where σ is the sp gr of the air

12. If w, w_1, w_2 are the weights required to sink a Nicholson's hydrometer in liquids of sp. gr s, s_1, s_2 respectively, prove that s can be determined in terms of the other quantities without knowing the weight of the instrument; further that this value of s does not require correction for the density of air

13 When the sectional area of the stem of a common hydrometer is non-uniform, find a mathematical expression for it so that the distances (along the stem) of the graduation marks should be in A P for values of the sp gr in A P

14. A man wishes to use a common hydrometer which is graduated for liquids lighter than water, for determining the sp. gr of liquids heavier than water. He attaches a weight at the lower end of the hydrometer and places the weighted instrument in milk (sp gr 1 05) and glycerine (sp gr 1 25), and notes that the respective readings correspond to specific gravities 0 63 and 0 77 When the instrument is put in nitric acid it indicates a sp gr 0 9, find the correct sp. gr of the acid

CHAPTER XI

HYDROSTATIC AND PNEUMATIC MACHINES

130. The machines which shall be discussed in this chapter are the siphon, the diving-bell, water- and air-pumps and the pressure-gauges

THE SIPHON

The siphon is used to empty a vessel containing some liquid without moving the vessel, the latter having no outlet at the bottom It consists of a bent tube ABC, open at both ends, one arm AB of which is shorter than the other arm BC To use the siphon, it is first filled with the same kind of liquid as is contained in the vessel (to be emptied), and both the ends A and C are closed The tube is then inverted and placed with the end A below the surface of the liquid in the vessel As soon as the ends are now opened, the liquid begins to flow in a continuous stream till the level of the liquid in the vessel falls below the end A, provided the vertical distance between A and the bend B be less than the height of the liquid barometer

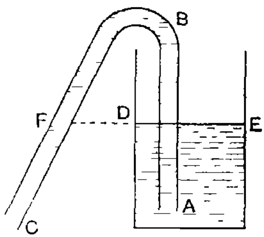

FIG 71

Just before the commencement of the flow, consider the forces acting on a very short length of the liquid at the end C Let a denote the cross-section of the tube, w the

weight of the unit volume of the liquid, Π the atmospheric pressure and x the vertical height of the level DE of the liquid (in the vessel) above the end C The forces acting on the element are: (1) the atmospheric pressure Πa on the end C, preventing the downward motion; (2) the pressure $(\Pi + wx)\,a$ on the upper side of the element, forcing it downwards, (3) its weight, and (4) the reaction of the sides of the vessel which does not prevent the downward motion of the element Thus, we see that on the whole the element is forced down When, as a consequence of this downward pressure, the liquid moves out of the tube, a partial vacuum is created near the bend causing the atmospheric pressure (acting on the surface of the liquid in the vessel) to force some liquid up the tube AB This action continues so long as A remains below the liquid

It is necessary (1) that the height of B above the liquid does not exceed the height of the barometer (of the liquid) for, otherwise, the atmospheric pressure would be unable to force the liquid up to the height B, (2) that the end C remains below the liquid-level in the vessel in order that the pressure on the element at C be downwards

The force of gravity helps the action of the siphon to continue automatically, the work done by gravity being that in bringing the liquid from the vessel to a lower level

THE DIVING-BELL

131. The diving-bell consists of a bell-shaped or cylindrical vessel made of metal and open at the lower end Its weight is sufficiently large to enable it to sink in water along with the air enclosed by it As the bell sinks, water rises within it, the greater the depth of the bell, the higher the level of water within, but in no case can water rise and fill up the whole volume In practice the bell is quite large so as to accommodate a number of men, for whom there is arrangement of seats like S, S'. It enables men

to go down to the bottom of deep waters, and to perform there such work as laying the foundation of a pier, etc The machine is lowered by a strong chain, K, attached to its top. There are also two tubes communicating the interior of the bell with the surface of the water, pure air can be pumped into the interior through one of the tubes to ensure a sufficient supply of oxygen and to keep down the water to any desired level, whilst the vitiated air is withdrawn from the bell through the other

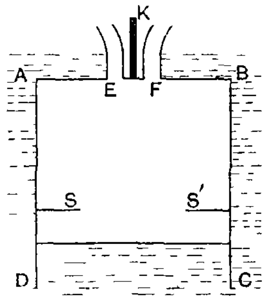

FIG 72

132. As the bell sinks down the pressure of the air within it increases, being equal to that at the level of the water inside the bell (Art 89) If the temperature remains constant—as is usually the case—the volume and the pressure of the enclosed air obey Boyle's law Thus, the volume of the air diminishes, *i e* the water rises in the bell, with the increase of the depth of the bell. When some more air is pumped from above, it should be noted, while applying Boyle's law, that the mass of the air within is increased

Let us next consider the equilibrium of the instrument in any position The forces acting on it are (1) the tension, T, of the chain which supports it, (2) the weight (including that of the air enclosed *), and (3) the fluid thrust, which is equal to the weight of the water displaced by the bell (together with the air)

* It is convenient to consider the equilibrium of the quantity of the air enclosed along with the bell Those who like to consider the bell alone should obtain the difference of pressure on the external and the internal sides of AB, taking into account pressures at different levels, even for the air

$\therefore\ T = $ wt of the bell + wt of the air enclosed

− wt of the water displaced .. .. (1)

As the last term diminishes in magnitude with the increase of the depth of the bell, it follows that the tension of the chain increases with the depth

133. *Ex* 1 A cylindrical diving-bell of height b is lowered into water till its top is at depth a below the surface To find the rise of water in the bell and also the amount of air that must be supplied from above so that there may be no water within the bell The height of the water barometer is h and the weight of unit volume of water is w

Let the top AA' of the bell $BAA'B'$ be at depth a below the surface LL', and let CC' denote the surface of water inside the bell Let $AC = x$, then $BC = b - x$ Also, let α be the cross-section of the bell

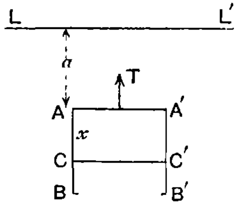

FIG 73

Pressure of the air CA'

$$= \text{pressure at the level } CC'$$
$$= wh + (a + x)\, w$$

Its volume $= x\alpha$, this quantity of air occupied a volume $b\alpha$ at the atmospheric pressure, wh Therefore, by Boyle's law,

$$x\alpha\ \ (h + a + x)\, w = b\alpha\ \ wh,$$

or $$x^2 + (h + a)\, x - bh = 0 \qquad (2)$$

This quadratic equation has obviously only one positive root, say β Therefore the rise of water $= BC = b - \beta$

Next, assume that a volume of air at atmospheric pressure which would have occupied a length y of this cylinder (*i e* a volume $y\alpha$ of air at pressure wh) be pumped in from above so as to expel all water from the bell Then the total mass of air (now filling the bell) occupied a volume

$(ba+ya)$ at pressure wh, its present volume is ba, and pressure = pressure at the level $BB' = wh + (a+b)w$

$\therefore$ by Boyle's law,

$$(ba+ya)\ wh = ba\ w(h+a+b),$$

or $$(b+y)h = b(h+a+b);$$

$$\therefore\ y = (a+b)\,b/h \quad \ldots \quad \ldots \qquad (3)$$

Lastly, substituting in (1), we get the tension of the chain at this depth, given by

$$T = W - wa\beta(1-\sigma), \qquad (4)$$

where σ is the density of the compressed air and W is the weight of the bell. But as the last term is very small as compared with the others, we can put

$$T = W - wa\beta \qquad (5)$$

Ex 2 Show that a cylindrical diving-bell whose cross-section is a, height b and weight Mg, will reach a position of equilibrium when put in water if

$$M < a\left[\frac{\sqrt{h^2+4bh}-h}{2} - b\varrho\right],$$

where h is the height of the water barometer and ϱ the density of the atmospheric air, the density of water being taken as unity. Also show that in the position of equilibrium $\dfrac{M}{a} + b\varrho$ is the depth of the level of water inside the bell below the external surface or below the top of the bell, according as the bell floats partially or wholly immersed in water. [Cf *Q* 20, *Examples* 12.]

As the bell is lowered, the air within it is compressed and water rises in the bell. The forces on the bell (and the air) in any position are (1) the weight Mg of the bell, together with the weight $bag\varrho$ of the air (because this fills the whole vessel at atmospheric pressure when its density

is ϱ), and (2) the upward fluid thrust, which is equal to the water displaced, there being no tension of the chain in this case. Obviously more and more water is displaced till the bell is completely immersed, and thereafter the amount of the displaced water will be less and less as the bell is lowered. Thus the maximum fluid thrust would be when the bell is just immersed. If the weight, $Mg + bag\varrho$, exceeds this value it is clear that the bell will sink under its own weight; on the other hand, if it be less than this value of the thrust there are two positions of equilibrium, one occurring before complete immersion and the other afterwards; fluid thrusts in both the positions being equal to $g(M + ba\varrho)$.

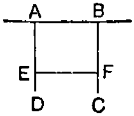

FIG. 74

To calculate the maximum fluid thrust, let AE (Fig. 74) be z; then the pressure of the enclosed air is $(z+h)g$ and its volume is za. Therefore, by Boyle's law,

$$za\,(z+h)g = ba\,hg,$$

or

$$z^2 + hz - bh = 0$$

This equation could have been deduced from (2) by putting $a = 0$. The positive root of this is

$$z_1 = \frac{\sqrt{h^2 + 4bh} - h}{2}$$

$\therefore$ fluid thrust is $z_1 ag$. Hence, in order that positions of equilibrium may exist,

$$g(M + ba\varrho) < z_1 ag,$$

whence the desired result follows.

For the position of equilibrium, let $z_2 a$ be the volume of water displaced.

$$\therefore\ g(M + ba\varrho) = z_2 ag, \text{ or } z_2 = \frac{M}{a} + b\varrho$$

Now, for the first position of equilibrium z_2 gives the difference between the levels of water inside and outside

the bell, in the second, it is the length of the air column in the bell

Ex 3 A diving-bell is suspended in water at a fixed depth, a solid body, which was resting on the platform (above the surface of water) inside the bell, suddenly falls into the water and floats freely within the bell Show that the level of water inside the bell rises, but the thrust of the chain and the amount of water in the bell are less than before.

Let v denote the volume of the object and σv that of its portion in water when it is floating freely. Also, let the volume of the bell originally unoccupied by water be V, then the initial volume of air $= V - v$

Now when the solid floats, if the level of water were the same as before, the volume of air would have been

$$V - (v - \sigma v) \text{ or } V - v + \sigma v,$$

i e greater than the original volume.

$\therefore$ by Boyle's law, its pressure would have been less than before So the level of water within the bell must rise

Next, let the subsequent volume of the air in the bell be V'. Then, since the pressure is less (as it has been proved that the water level is higher), final volume $>$ initial volume or $V' > V - v$

The original volume of water in the bell

$$= \text{total volume of the bell} - V,$$

and the final volume of water in the bell

$$= \text{total volume of the bell} - (V' + v)$$

But $V' + v > V$, proved before Therefore the subsequent volume of water in the bell is less than the original volume

Lastly, by formula (1), the initial tension, T,

$= $ sum of wts of bell, solid and air $-$ wt of vol V of water displaced.

and the final tension,* T',

$$=\text{sum of the wts of bell, solid and air} - \text{wt of vol } (V'+v) \text{ of water displaced}$$

It follows that $T' < T$

EXAMPLES. 17.

1. A diving-bell is lowered in a lake until one-half of it is filled with water · prove that if d be the depth of the top of the bell below the surface, the height of the bell is $2(h-d)$, where h is the height of the water barometer

2. When a pin-hole is made on the top of a diving-bell immersed at some depth, it is observed that air escapes from the bell. Explain the reason

3. A diving-bell is lowered in water until one-fourth of it is occupied by water ; d is the depth of the top of the bell and h the height of the water barometer. Find how much air at atmospheric pressure must be pumped in from above so that (i) there may be no water inside, (ii) the quantity of water in the bell be reduced to half

4. A diving-bell of capacity 1320 cu in. and originally full of air at atmospheric pressure and at 12° C, is lowered into water which is at 31° C, till its lower edge is 10 2 ft below the surface. How many cu in of air at atmospheric pressure and at 12° C must be pumped into the bell so that when the contained air shall have acquired the temperature of water it may just fill the bell ? The water barometer stands at 33 ft

5 The water inside a conical diving-bell stands at $\frac{2}{7}$ths of the height of the bell ; the atmospheric temperature is 42° C, while the temperature inside the bell is 27° C Prove that the depth of the water level inside the bell below the surface is $\frac{121}{75}$ times the height of the water barometer

6. A hemispherical diving-bell (of radius r) is lowered in water with its base horizontal till the water rises up to the middle point of the vertical radius of the bell Show that the

* If we leave out the solid from the equation, we get the fluid thrust $=$ wt of the vol $V'+(v-\sigma v)$ of water, $v-\sigma v$ being the part of the solid above water But wt of the vol σv of water equals the wt. of the solid Thus, the fluid thrust $=$ wt of vol $(V'+v)$ of water $-$ wt of the solid, and $T' =$ wt of bell (plus air) $-$ fluid thrust $=$ the same value as given above.

depth of the base of the bell below the surface is $\frac{11}{5}h + \frac{r}{2}$, h being the height of the water barometer.

7. If h, h' be the heights of mercury in a barometer placed within a cylindrical diving-bell of length b, at the beginning and the end of a descent, H the height of a mercury barometer at the surface and ρ be the sp gr of mercury, find the amount of the descent

8. A diving-bell having the form of a cylinder surmounted by a hemisphere, the height of the cylinder being equal to the radius of the common base, is lowered in the sea until the water occupies the whole cylinder The height of the water barometer is 32 ft. and the sp gr of sea-water is 1 025 Find the depth of the water-level inside the bell below the surface of the sea

9. A small piece of wood floats with two-thirds of its volume immersed in water exposed to atmosphere If this piece be introduced in a diving-bell which is lowered to such a depth that the bell is half filled with water, find what fraction of the solid will be immersed when floating freely Sp gr of the atmospheric air is ρ

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10 A diving-bell in the shape of a paraboloid of revolution of altitude b, is lowered into water with its axis vertical Prove that (1) the pressure of the air within the bell varies inversely as the square of the portion of the axis unoccupied by water, (2) the height of the water barometer is $\frac{(n-1)^2}{n} b$ if, when the bell is lowered till its vertex is at a depth b, the water occupies $\frac{1}{n}$th part of the axis

11 A small hollow sphere made of thin elastic material contains air of three times the external density when exposed to atmospheric air. It is suspended within a cylindrical diving-bell of height 8 ft , which is then sunk to a depth x ft below the surface of water , it is found that the sphere is now one-tenth less in volume than when the bell was at the surface Find x, on the supposition that the force of compression due to the tension of the elastic envelope remains practically unaltered. The height of the water barometer is 33 ft

12. A cylindrical diving-bell is lowered in fresh water by means of a chain so that its top is at a certain depth below the surface. If it be lowered in sea to the same depth, prove that the tension of the chain is less than before.

THE COMMON PUMP

134. The principle of suction is used in the construction of pumps A partial vacuum is created in a space and the external pressure, being far in excess over the internal, forces the external fluid in to fill up the space.

Valves are used in all pumps, their object is to allow the fluid to pass through in one direction and to prevent any fluid to pass through in the opposite direction. Generally, it consists of a smooth piece of metal (or leather or oiled silk) closely fitting a hole, when a force is applied, no matter how small, it moves and exposes the hole through which the fluid passes, but if the force is applied in the opposite direction it shuts the hole too closely to allow any fluid to pass through The action of a valve in practice, however, is not so perfect, a definite amount of pressure is required before it opens, and there is always some amount of leakage.

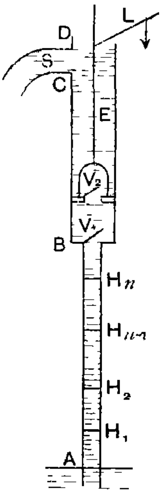

FIG 75

135. The **common** *or* **suction pump** is used to transfer water from one place to another at a higher level It consists of a long cylinder AB which dips into the reservoir from which water is to be raised, at the upper end B, it is connected with another cylinder BD of wider cross-section, into which a piston E works between B and the point C where the spout S of the pump is attached The piston and the junction B (of the two cylinders or barrels) are fitted with valves V_2 and V_1, *both opening upwards* The piston rod is worked by a lever L.

To explain the *action of the pump* let us suppose that in the beginning the piston is at its lowest position, viz B,

and that the water has not risen in the tube AB Let the piston be moved upwards. As it moves a partial vacuum is created between it and B, the air below B being at greater pressure forces the valve V_1 upwards, and some air passes from the lower cylinder to the upper If the valves be perfect and the piston air-tight, as we shall assume to be the case in our explanation and calculations, the quantity of air which was confined in AB, occupies greater and greater volume with the ascent of the piston, and so its pressure (and density) becomes less and less Consequently the atmospheric pressure on the external surface at A forces some water into the barrel AB up to a certain height, H_1 (say), when the piston reaches C During this half-stroke of the piston, the valve V_2 remains shut, as the pressure on its lower face is less than that on the upper, viz the atmospheric pressure

Next, the piston is pressed down, the air between B and C is compressed, its pressure increases and shuts V_1 Thus the air between B and H_1 remains at the same pressure as when the piston was at C After some time the pressure of the air between B and the piston exceeds the atmospheric pressure and so opens the valve V_2, therefore, when the piston reaches B, all the air in the upper barrel below the piston has escaped Thus the effects of one complete stroke of the piston are (1) that water rises in the lower cylinder (up to H_1), and (2) some air, originally in this barrel, has escaped and there remains a smaller quantity at a lesser pressure (whose value is equal to that at the level H_1)

Similar actions take place in every successive stroke of the piston, after the second stroke water rises to a greater height H_2, the air in the lower cylinder being less in quantity and at a smaller pressure than before In this manner water rises higher and higher in the lower cylinder until during the first half of the subsequent stroke water enters the

upper cylinder When the piston is brought down to B, all the air (below the piston) is expelled, there being now some water above the piston itself At the next stroke the water above the piston is carried upwards and escapes through the spout, whilst more water (from the lower barrel) follows the piston up to C, provided its height above A does not exceed the height of the water barometer

The success of the pump depends on the atmospheric pressure being able to force water into the upper cylinder at least, if not up to the level C The height AB of the lower cylinder should, therefore, be less than the height of the water barometer

136. *To find the distance through which the water rises during the nth stroke of the piston, assuming that there is some air in the lower cylinder at the beginning of the stroke*

Let H_{n-1} and H_n (Fig 75) denote the levels of water at the beginning and the end of the nth stroke Let

$$AH_{n-1}=x_{n-1},\quad AH_n=x_n,$$

$AB=a$, $BC=b$, the cross-section of the lower cylinder $=\alpha$, that of the upper $=\beta$, the height of the water barometer $=h$ and the weight of unit volume of water $=w$ We shall suppose that the volumes of the valves and the piston are small compared with the capacities of the cylinders

Firstly, let H_n be in the lower cylinder At the beginning of the stroke the volume of air in this cylinder was

$$(a-x_{n-1})\,\alpha,$$

and its pressure was $w(h-x_{n-1})$ After the upward stroke, this air occupied the volume $=(a-x_n)\,\alpha+b\beta$ and its pressure $=w(h-x_n)$, viz that at the level H_n Therefore from Boyle's law, we have, after removing the common factor w,

$$[(a-x_n)\,\alpha+b\beta]\,(h-x_n)=[(a-x_{n-1})\,\alpha]\,(h-x_{n-1}) \quad \ldots (6)$$

From this equation x_n would be obtained if x_{n-1} is known, and thence the value of $x_n - x_{n-1}$ or the rise in the water-level during the nth stroke

At the beginning of the first stroke, water was at the level A and at the commencement of the succeeding strokes it was at H_1, H_2, . , let $AH_1 = x_1$, $AH_2 = x_2$, We can obtain, like (6),

$$[(a - x_1)\,\alpha + b\beta]\,(h - x_1) = \alpha a h, \tag{7}$$

$$[(a - x_2)\,\alpha + b\beta]\,(h - x_2) = (a - x_1)\,\alpha\,(h - x_1),$$

and so on These equations can be obtained from (6) by putting $n = 1, 2,$ and taking $x_0 = 0$ The equation (7) gives x_1, the next gives x_2, as x_1 is now known, and so on

Secondly, let the level H_n be in the upper cylinder At the end of the upward stroke water rises to H_n, therefore the volume of air $= (a + b - x_n)\beta$ at pressure $w(h - x_n)$ Hence we shall have

$$(a + b - x_n)\beta\,(h - x_n) = (a - x_{n-1})\,\alpha\,.\,(h - x_{n-1}), \tag{8}$$

giving x_n During the next stroke all air is expelled and water passes out of the spout as has already been stated

137. The **lifting pump** is a modification of the common pump and is used to lift water to a high level, the distance depending on the strength of the pump

In this pump the top of the upper cylinder is closed and the piston rod, T, works through a hole (in the lid), which it fits closely so as to be air- and water-tight The spout is fixed to the cylinder just below its top and consists of a long vertical tube of sufficient length with an outlet above, there is an additional valve, V_3, fitted at the neck of the spout, opening outwards from the cylinder

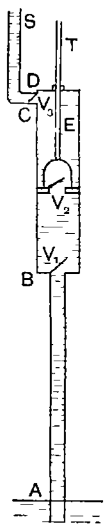

FIG 76

The behaviour of the air and water below the piston is the same as in the common pump We have, therefore, to consider the action above the piston As it is moved upwards during the first stroke, the air between it and D is compressed and so opens the valve V_3, whilst the valve V_2 is kept shut When the piston arrives at D all the air (in the barrel) above it has been expelled through the spout During the downward motion vacuum is created above the piston, therefore V_2 opens and the air in the upper cylinder passes from below the piston to the upper region, the valves V_3 and V_1 remaining closed This action is repeated during several succeeding strokes till water enters the upper cylinder and remains above the piston when the latter is at B During the next upward stroke all air is expelled and water enters the tube CS, its level rising higher and higher at every subsequent stroke

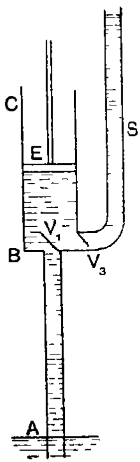

FIG 77

138. The **forcing pump** is another modified form of the common pump, and enables water to be forced up to any height provided that the machine is strong enough

In this pump there is no valve on the piston, which has a solid base fitting the cylinder BC tightly The spout is fixed at the bottom B, and consists of a long vertical tube as in the lifting pump, having a valve V_3 at its mouth opening inwards

We need not consider the action in the lower barrel AB, which is the same as described in Art 135 During the downward stroke the piston drives the air (or water, or both as the case may be) between itself and B, through the valve V_3 into the tube S, the valve V_1 remaining closed during this action During

the upward motion, V_1 opens while V_3 remains closed, thus allowing air or water from AB (but not from S) to enter the upper barrel.

139. There will be flow through the tube S of the forcing pump during the downward stroke only of the piston In order that the flow be continuous the water is first led from the upper cylinder (which is not completely shown in Fig 78) into a chamber V partially filled with water The spout S' leads water upwards from this chamber to the required height, its lower end being at a sufficiently low position within the chamber

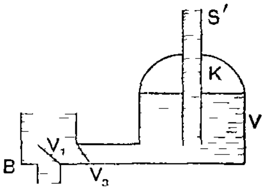

FIG 78

When the piston moves downwards the water is forced into the chamber and then into the tube S' So there is flow of water during this half-stroke, the air in the chamber is also compressed at the same time When the piston moves upwards, the pressure caused by the piston is released, the volume of compressed air tries to expand and thus forces some water (from the chamber) up the tube S'.

140. The **fire-engine** is essentially a forcing pump with an air-chamber described in the last article There are, however, two upper barrels, each being provided with two valves such as V_1 and V_3, and a piston working in it The two barrels communicate with each other below the valves (V_1) so that the machine can work with one lower cylinder only. The pistons are worked in such a manner that when one goes up the other moves down

141. *To find the tension of the piston rod.*

Assuming that the piston moves uniformly during the major part of a half-stroke, we see that the tension of the

rod is such (in magnitude and direction) that it would balance the pressures on the upper and the lower sides of the piston Even this value is only approximate, because the effects of the forces like gravity, friction of the sides of the barrel are not taken into account

For example, consider the value of the tension when the piston moves upwards, there being some air in the cylinders Let the water level be at H_n and $AH_n = x_n$ (Fig 75)

Pressure of the air on the under side of the piston

$$= a' \times \text{pressure of the air underneath}$$
$$= a' \; w(h - x_n),$$

where a' denotes the area of the piston.

And pressure on the upper side

$$= a' \; wh$$

$\therefore$ the tension is upwards and $= a' \; wx_n$ in magnitude.

THE AIR-PUMP

142. Air-pumps are used either to remove air from a vessel or to add some air to it Those belonging to the latter class are generally called *condensers*, whilst the name air-pump is restricted to those of the former type

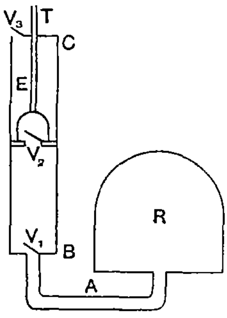

FIG 79

Smeaton's air-pump consists of a cylinder CB in which a piston, E, works, there are three valves all opening upwards, V_1 at the base and V_3 at the top of the barrel CB, the third, V_2, being fitted to the piston Below B runs a tube A which connects the pump to the vessel (or *receiver*) R which is to be emptied of air The piston rod works through an opening at the top of the barrel, which it fits closely The different parts of the instrument are made as air-tight as possible

The range of the piston, *i e* the distance through which it moves up and down, is generally taken, for calculations, to be the whole length of CB, though in practice it is a little less, there being some "clearances" left on both sides.

Suppose the pump commences working when the piston is at its lowest position, viz at B As it is moved up, the partial vacuum created between the piston and the end B causes the valve V_1 to open, the air in the receiver and the tube A expands occupying the additional volume BC when the piston reaches its highest position, viz C It is clear that this quantity of air is now at a common reduced pressure (or density) During this operation the air above the piston is compressed, therefore the valve V_2 remains closed, while V_3 opens and lets out the air above the piston

During the downward stroke vacuum is created between C and the piston, consequently the air underneath forces V_2 open and passes above When the piston reaches B all the air in the cylinder has passed (through V_2) above the piston The valves V_3 and V_1 remain shut, since, in the first case the atmospheric pressure is greater than that within and, in the second, the air between B and the piston (being compressed) is at a greater pressure than the air in the receiver

Thus, the effect of one complete stroke of the piston is that the pressure (and the density) of the air in the receiver is reduced ; the volume of air now occupying R is the same as before and therefore its mass must be less This is repeated at every succeeding stroke till the pressure in the receiver is sufficiently reduced, or is diminished to such an extent that it is unable to lift the valve V_1

143. Hawksbee's air-pump (single-barrelled) is the same as the preceding, except that the top of the barrel BC is

kept open. Its action can easily be explained. This pump is also constructed with two barrels (like the fire-engine), the piston working in the one going down whilst the other is moving up.

The advantages of Smeaton's pump over a pump of this type are (1) that the pressure above the piston in the former pump, being much less than the atmospheric pressure which acts on the upper surface of the piston of the latter, makes it easier, after a few strokes, to lift the piston during the greater part of the upward stroke, and (2) that for the same reason the valve V_2 opens much more readily in the former pump than in the latter Thus the exhaustion of the receiver can be carried to a greater extent by means of the Smeaton's pump

144. *To determine the density (and pressure) of the air in the receiver of an air-pump (Smeaton's or Hawksbee's) after n complete strokes of the piston*

Let V be the volume of the receiver,* v that of the barrel BC, ϱ the density of atmospheric air (i e initial density of air in the receiver), and $\varrho_1, \varrho_2, \ldots \varrho_n \ldots$ denote the densities of air in R after $1, 2, \ldots n, \ldots$ complete strokes.

After the first upward stroke, the volume V of the air in the receiver has increased to $V+v$

$$\therefore \text{ its mass} = V\varrho = (V+v)\varrho_1,$$

or

$$\varrho_1 = \frac{V}{V+v} \cdot \varrho$$

During the downward stroke the density of the air in the receiver is unaffected After the second upward stroke

* Strictly speaking, this should include the volume of the air in the connecting tube A But the latter is generally very small as compared with the capacity of the receiver , hence its effect may be neglected in practice

the volume V of air in the receiver (of density ϱ_1) has increased to $V+v$

$$\therefore \text{ its mass} = V\varrho_1 = (V+v)\varrho_2,$$

or
$$\varrho_2 = \frac{V}{V+v}\varrho_1 = \left(\frac{V}{V+v}\right)^2 \varrho$$

And so on In this manner we shall obtain

$$\varrho_n = \left(\frac{V}{V+v}\right)^n \varrho \qquad (9)$$

It follows that the subsequent pressure p_n (in the receiver)

$$= \left(\frac{V}{V+v}\right)^n \Pi, . \qquad (10)$$

where Π is the atmospheric pressure [Cf Art 101]

The result (10) can also be obtained independently Let $p_1, p_2, \ldots p_n$. denote the pressures in the receiver after 1, 2, n . strokes and let the original pressure be Π By Boyle's law,
$$\Pi V = p_1(V+v),$$

or
$$p_1 = \frac{V}{V+v}\Pi$$

And so on as before Thus

$$p_n = \left(\frac{V}{V+v}\right)^n \Pi$$

The forms of ϱ_n and p_n show that these would be zero if n be infinite, *i e* even theoretically a perfect vacuum cannot be obtained by this pump

145. Tate's air-pump consists of two pistons D and E (Fig 80) rigidly connected together, which move up and down in a cylinder (or barrel), CAB, being worked by a rod R passing through an air-tight collar at C There are two valves, V_1 and V_2, both opening outwards At the middle of the cylinder is a tube A which joins the pump to the receiver The distance DE is slightly less than half the length of the barrel, so that when E is at B, D is just below

the passage A. When the pistons are moved up to C the air between D and C is driven out through V_2, there being vacuum below E till this part of the barrel is brought into communication with the receiver, thus allowing the air in the receiver to expand and fill the space EB as well. The motion of the pistons is then reversed, the air between E and B is driven out through V_1 and the remaining air in the receiver expands into the upper half of the barrel (where a vacuum was created). In this manner by every upward or downward motion of the pistons the exhaustion is carried on.

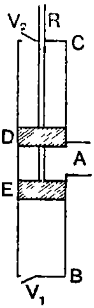

FIG. 80.

It should be noted that this air-pump contains neither the piston-valve nor the valve leading to the receiver; for this reason it can produce a greater degree of exhaustion than Smeaton's air-pump.

146. None of the aforesaid pumps can produce a very high degree of exhaustion as is required in the globes of electric lamps or some other apparatuses. It then becomes necessary to use a pump such as **Sprengel's air-pump**. After the vessel is exhausted to a certain extent with the help of the ordinary air-pumps, it is connected to this air-pump for further exhaustion.

This pump consists of a long vertical tube having a cup at its upper end and a stop-cock a little below the cup. Its lower end dips into mercury contained in a vessel. At a short distance below the stop-cock the tube branches out into another tube leading to the receiver. The length of the vertical tube is such that the height of the junction (with the second tube) above the mercury in the vessel is greater than the height of the mercury barometer. The cup is also filled with mercury, and the liquid is allowed to run down the tube. At the junction there is air which

comes from the receiver, and as mercury passes through this junction it breaks into drops, encloses a little air underneath it (below this point) and carries the same along these drops of mercury are separated from one another by small bubbles of air which escape into the atmosphere through the external surface of the mercury in the vessel The air from the receiver is thus withdrawn The upper cup is never allowed to be empty

Mercury rises in the tube as the pressure in the receiver diminishes When the exhaustion is complete, mercury drops do not carry air along with them and so fall with a metallic sound

147. The **Air-condenser** consists of a cylinder AB (open at the top) into which works a piston E At the end B of the cylinder is a valve V_1 which opens outwards The piston is supplied with another valve V_2 opening downwards The pump is connected to the vessel D (or the receiver), into which air is to be pumped, by a tube below the valve V_1, which is fitted with a stop-cock S

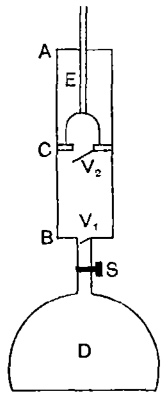

FIG 81

Suppose the receiver D to be full of atmospheric air and the piston at its lowest position, viz B When the piston is moved upwards V_1 shuts, but V_2 opens owing to the creation of a partial vacuum between B and the piston Thus no air from D can escape (even when the stop-cock is open), whilst the atmospheric air from above fills the space below C. When the piston reaches A, its highest position, the barrel from B to A is full of air at atmospheric pressure During the downward stroke this volume of air is compressed, causing the valve V_1 to open and V_2 to remain closed This volume

of air is completely forced into the receiver when the piston comes again to B Thus, in every complete stroke of the piston, a quantity of air, which would fill the cylinder AB at atmospheric pressure, is forced into D

148. *To find the density (and pressure) of the air in the receiver after n strokes of the piston*

Let V and v be the volumes of the receiver and the cylinder AB, also let ϱ, ϱ_n be the densities and Π, p_n be the pressures of the atmospheric air and the air in the receiver (after n strokes) In n strokes a quantity of air has been added to the original quantity in the receiver, which would occupy a volume nv at the atmospheric pressure Therefore the mass of the additional air is $nv\varrho$, and the total mass is $(V+nv)\varrho$ But this occupies now a volume V and its density is ϱ_n.

$$\therefore \quad V\varrho_n = (V+nv)\varrho,$$

or

$$\varrho_n = \frac{V+nv}{V}\varrho. \qquad (11)$$

It follows that

$$p_n = \frac{V+nv}{V}\Pi \qquad (12)$$

The last result can also be deduced in another way Let $p_1, p_2, \ldots$ be the pressures in the receiver after 1, 2, .. strokes. At the end of the first upward stroke the volume of the air in the receiver and the barrel was $V+v$ and its pressure Π After the first stroke this quantity occupied a volume V Therefore its pressure p_1 is given by

$$p_1 V = (V+v)\Pi \text{ or } p_1 = \frac{V+v}{V}\Pi.$$

At the end of the second upward stroke, the volume of the air in the receiver was V at pressure p_1 and that in the barrel was v at pressure Π After the completion of the second stroke, this quantity of air occupied a volume V at pressure p_2

$\therefore$ by (23) of Art 106,

$$p_2 V = p_1 V + \Pi v = (V + v)\Pi + \Pi v$$

$$\therefore\ p_2 = \frac{V + 2v}{V}\Pi$$

And so on

149. Manometer or pressure-gauge. In Art 97 has been described a manometer for measuring pressures It is obvious that to measure very low pressures the length of the arm BC (see Fig 63) should be slightly greater than the barometric height, and that to measure high pressures even this length would not suffice A modification of the apparatus is therefore used for high pressures The end D is closed so that there is some air above mercury whose volume (or length if the tube is uniform) at atmospheric pressure is known When A is joined to a receiver containing gas at a high pressure, mercury in BC falls to E (say), whilst that in CD rises up to F Now the pressure of the enclosed air in the tube CD can be calculated by Boyle's law, since the length FD (and therefore the volume) of the air is known, let this pressure be p Then the pressure in the receiver

$=$ the pressure at the level E

$=$ pressure at the level F + wt of the column EF

$= p + w \cdot EF$,

w denoting the weight of unit volume of mercury

A horizontal tube (closed at one end) with a drop of mercury enclosing a given amount of air near the closed end is sometimes used as a pressure-gauge The open end is connected with the receiver when the drop moves, compressing (or otherwise) the enclosed air In the position of equilibrium of the drop the pressure on its two ends is equal, whence the pressure in the receiver can be calculated

Low pressures are also measured by a *siphon gauge*, which is the same as described in Art 97, with the exception that the end D is closed and that there is vacuum above mercury in the tube CD

150. *Ex* 1 In a common pump, if the water just enters the upper cylinder during the second stroke, prove that

$$h^2\left(1-\frac{a}{nb}\right)\left(2-\frac{a}{nb}\right)-h\left(4a+nb-\frac{3a^2}{nb}\right)+a(nb+2a)=0,$$

where the letters have the same meaning as in Art 136, and $n=\beta/a$

Let x_1, x_2 denote the heights to which water is raised during the first and the second strokes; here x_2 is given to be a. From the equation (7) and the next of Art 136, we get

$$[(a-x_1)+nb](h-x_1)=ah, \qquad \text{(i)}$$

and

$$nb(h-a)=(a-x_1)(h-x_1) \qquad \text{(ii)}$$

From (i) and (ii), we get

$$nb(h-a)+nb(h-x_1)=ah,$$

or

$$h-x_1=\frac{ah}{nb}-(h-a),$$

$$\therefore\ a-x_1=\frac{ah}{nb}-2(h-a)$$

Substituting these values in (ii) we get

$$nb(h-a)=\left[\frac{ah}{nb}-(h-a)\right]\left[\frac{ah}{nb}-2(h-a)\right],$$

whence the desired result follows after expansion and rearrangement

Ex 2 If c be the length of the barrel in a Smeaton's air-pump, a the distance of the piston in its highest position from the top of the barrel, b the distance in its lowest position from the base of the barrel, and Π the atmospheric

pressure, show that the pressure of the air in the receiver cannot be less than

$$\frac{ab}{(c-a)(c-b)}\Pi$$

Let DL denote the range of the piston, then $BL=b$, $CD=a$ Let the cross-section of the barrel be α, the volume of the receiver be V, the pressure of air in it be p and that of the air in the barrel be p_1 after some strokes, also let the piston be at L at that time It is clear that pressures of air on both sides of the piston are the same, since during the preceding downward motion the valve V_2 was open, establishing communication between the two parts

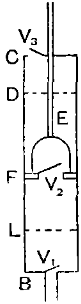

FIG 82

During the next upward stroke *let the valve* V_1 *open*,* so that the air in the receiver and the barrel (below the piston at D) assumes a pressure p' ($<p$)

by (23) of Art 106,

$$pV+p_1b\alpha=[V+(c-a)\alpha]p'$$

Since $pV>p'V$, we have

$$p_1b\alpha<(c-a)\alpha p',$$

or $$p'>\frac{b}{c-a}p_1 \qquad (1)$$

At the same time the air above the piston is compressed, let the compression be *sufficient to open the valve* V_3, thus allowing some air to escape and the remainder (which occupies the length CD or a) to assume the atmospheric pressure Π Since the amount of air is less than before (cf Art 101),

$$p_1(c-b)\alpha>a\alpha\ \Pi,$$

* The opening of the valves is necessary for the proper working of the pump, *i e* for the further exhaustion of the receiver

or $$p_1 > \frac{a}{c-b} \cdot \Pi \quad .. \qquad \text{(ii)}$$

$\therefore$ from (i), $$p' > \frac{ab}{(c-a)(c-b)} \Pi .$$

This shows that the pressure in the receiver cannot be less than the right-hand side It follows that the density of the air within the receiver cannot be less than $\frac{ab}{(c-a)(c-b)}$ times the atmospheric density

Ex 3 In a common pump, if the piston does not go home up to the lower valve of the barrel, prove that no water can in general be lifted into the upper cylinder unless the range of the piston is greater than $\frac{ab}{h}$, where a, b, h have the same meaning as in Art 136

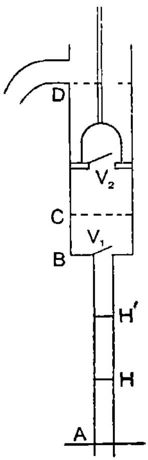

FIG 83

Let c denote the distance between the lowest position C of the piston from the lower valve *i e* let $BC = c$, also let H be the water level at the beginning of a certain stroke, AH being x The volume of the air from H to B is at pressure $(h-x)\,w$ and the volume from B to C at hw, since during the previous downward stroke the piston valve was opened allowing communication between the air below the piston and the atmosphere

Let the valve V_1 open (see footnote, Ex 2) during the next upward stroke, the above-mentioned volumes of air expand and occupy the volume from D to the water level, which is now at H' (higher than H) Its pressure is now $(h-x')\,w$ where $AH' = x'$

$\therefore$ by (23) of Art 106, removing the common factor w,

$$(h-x) \,.\, (a-x)\,\alpha + h \quad c\beta = (h-x') \quad [(a-x')\,\alpha + b\,\beta] ,$$

α, β, w are as in Art. 136.

But $$(h-x)(a-x)\alpha > (h-x')(a-x')\alpha,$$
since $x' > x$. Therefore
$$hc\beta < (h-x')b\beta,$$
$$\therefore\ x' < \frac{h(b-c)}{b} \text{ or } < \frac{hy}{b}, \quad \ldots \ldots \ldots \text{(iii)}$$

where $y = b - c = CD$ denotes the range of the piston This shows that water cannot rise higher than hy/b, in order that water may (at least) just enter into the upper cylinder we must have
$$\frac{hy}{b} > x' \text{ or } a, \text{ whence } y > \frac{ab}{h}.$$

EXAMPLES. 18.

1. The lower valve of a common pump is 27 ft above the surface of water in a well, and the area of the cross-section of the upper cylinder is twice that of the lower If the length of the upper cylinder be 38 ft, find how much water rises at the end of first and second strokes, the height of the water barometer being 34 ft

2. If the common pump of the last example be used to raise petroleum (sp gr 0 85) from a well, find how much liquid is raised at the end of the first stroke

3. Water is to be pumped up from a well in which there is water at a depth of 10 ft from the ground-level. The base of the upper barrel of the pump is at a distance of 2 ft above the ground, the length of each stroke of the piston is 1 ft, the ratio of the cross-sections of the two barrels is as 12 · 1, and the atmospheric pressure is equal to the weight of a column of 32 ft of water Show that water enters the upper barrel before the completion of the second stroke

4. Find the ratio of the volumes of the receiver and the barrel of a condenser if, after nine strokes, the pressure in the receiver increases to four times its original value.

5. If the volume of the receiver be ten times as large as that of the barrel of an air-pump, find the number of strokes that must be made before the pressure inside the receiver becomes less than one-fourth of its initial value. Given log 2 = 3010, log 11 = 1 0414.

6. In a Smeaton's air-pump, find the position of the piston during the $(n+1)$th ascent when the highest valve just begins to open. The weight of the valve is to be neglected.

7. If the upper valve of a Smeaton's air-pump, which is of negligible weight, opens when the piston is $\frac{7}{8}$ths of the way up, find the density of the air in the receiver before the ascent.

8. The mass of the air in the receiver of an air-pump is m at the beginning, and it becomes m' after n complete strokes of the piston. If V and v denote the volumes of the receiver and the barrel, prove that

$$n = \frac{\log m - \log m'}{\log (V+v) - \log V}.$$

9. A condenser and a Smeaton's air-pump have equal barrels and a receiver is simultaneously connected to both, the volume of which is 20 times that of each barrel. If the condenser be worked for 20 strokes and then the pump for 14, show that the density of air in the receiver is nearly the same as originally Given that log 2 = 3010, log 21 = 1 3222.

10. Air is being uniformly forced into a receiver by a condenser, the receiver is fitted with a gauge which consists of a drop of mercury in a fine horizontal tube closed at the farther end. If A be the position of the mercury drop when the air was uncompressed, B the closed end of the tube and C the position of the mercury after some complete strokes, prove that the ratio AC CB increases in an arithmetic progression.

11. If the piston of a Smeaton's air-pump can move through a length c of the cylinder while there are clearances at the top and the base of lengths a and b respectively, find the pressure, after two strokes, of the air in the receiver (whose volume is n times that of the barrel

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12. A cylindrical vessel whose height equals that of a water barometer is three-quarters filled with water and fitted with an airtight lid. If a siphon be inserted through a hole (in the lid) which the siphon fits closely so as to be watertight, so that its highest point is in the surface of the lid, and the end of its longer arm is on a level with the bottom of the vessel, prove that one-third of the water may be removed by the action of the siphon

13. Show that the pressure p of the receiver is given in terms of the height x of mercury in a compressed air manometer by the relation

$$p = wx\left(1 + \frac{k}{K}\right) + P \cdot \frac{a}{a-x},$$

where a is the height of the tube containing compressed air, k its cross-section and x the height of mercury in it; also, K is the cross-section of the other tube leading to the receiver, P the atmospheric pressure and w the weight of unit volume of mercury. The quantities a and x are measured upwards from the point at which the mercury level in the former tube (of the manometer) stands when the instrument is exposed to the atmospheric pressure

14 The volumes of the receiver and the barrel of a Hawksbee's air-pump are V and v respectively, and P is the atmospheric pressure, prove that the work done in slowly raising the piston during the $(n+1)$th stroke is

$$P\left[v - \frac{V^{n+1}}{(V+v)^n}\log\left(1+\frac{v}{V}\right)\right],$$

the forces of gravity being neglected

15. In a condenser the piston can only work through a length a, the distance of the piston in its lowest position from the lower valve being b Show that the density of the air in the receiver cannot exceed $\frac{a+b}{b}$ times the atmospheric density

16 In a Hawksbee's air-pump, A and B denote the volumes of the receiver and the barrel and C the volume of the part of the latter untraversed by the piston (*i e* the clearance is of volume C) Show that the density of the air in the receiver after n strokes of the piston is

$$\frac{C}{B} - \left(\frac{C}{B} - 1\right)\left(\frac{A}{A+B}\right)^n$$

times the original (*i e* atmospheric) density

17. If the piston in a Smeaton's air-pump does not traverse the whole length of the barrel, but leaves a clearance of volume C at the bottom and another of volume C' at the top, show that the density in the receiver after n strokes of the piston is given by

$$\varrho_n - \mu^n\varrho = \frac{CC'}{(B-C)(B-C')}(1-\mu^n)\varrho,$$

where

$$\mu = \frac{AB + BC - CC'}{B(A+B-C')},$$

A, B being the capacities of the receiver and the barrel respectively and ϱ the density of the atmospheric air Deduce the limiting value of the density

18. In a Smeaton's air-pump show that the tension in the piston rod at the instant when the uppermost valve begins to open during the $(n+1)$th upward stroke, is

$$\frac{(1-\mu^n)a}{1-\frac{\mu^{n+1}v}{V}}$$

times the atmospheric pressure where V, v denote the volumes of the receiver and the cylinder, a the cross-section of the piston and $\mu=\frac{V}{V+v}$.

ANSWERS AND HINTS

EXAMPLES. 1.

1. Not more than $345\frac{3}{5}$ ft.

2. $2093\frac{1}{4}$ lbs weight

4. $\dfrac{mp' + np}{m+n}$.

6. $\frac{1}{2}g\varrho\, dn(n+1)$. For the second part put $\varrho = \lambda d$ where λ is some constant before proceeding to the limit.

8. Surface of separation is on a vertical arm at a height $1\frac{7}{8}$ in.

9. $3\frac{7}{68}$ in., $17\frac{61}{68}$ in

10. The heavier liquid occupies the cylinder of height a, and a length $\dfrac{a\varrho - b\varrho'}{\varrho - \varrho'}$ of the other

12. $\tan^{-1}\frac{1}{2}$; $\tan^{-1}2$.

13 $\tan^{-1}[\frac{1}{2}(8+\sqrt{2})]$

14. The pressure must balance the component of the weight

15. Utilise the property that the product of the abscissae of the extremities of a focal chord $= a^2$, $4a$ being the latus rectum.

EXAMPLES. 2.

1. 2,500,000 lbs weight.

2. $601\frac{9}{16}$ lbs weight; $605\frac{15}{32}$ lbs weight

3. The ratio $= 3:1$

4. $10\ 96wah^3$, where a is the breadth, $3h$ is the depth of the vertical side and w is the weight of unit volume of water.

6. $\frac{7}{4}$ times the original pressure.

7. The maximum (or minimum) thrust occurs when the centre of gravity, G, is vertically below (or above) A The ratio $= h + AG \quad h - AG$.

8. $\dfrac{12\tan^2 a}{\sqrt{16\tan^2 a + 1}}$ times the weight of the water contained

9. $\frac{12}{\sqrt{73}}$ times the weight of the liquid contained.

10. Let A be the vertex and BC the base, from AB cut off $AD = \frac{1}{\sqrt{2}}$ of AB, and draw DE parallel to BC.

13. Since the diagonal AC divides the area into two equal triangles, the thrust on the lower is clearly greater than that on the upper, the required line must be below AC. From DC cut off $DE = \frac{1}{4}DC$, join AE.

14. On each of the upper faces, $\frac{g\varrho a^3}{24}\sqrt{6}$, on each of the lower faces $\frac{g\varrho a^3}{12}\sqrt{6}$.

15. The chord will be $\frac{4}{5}$ of the radius below the centre, $50\sqrt{5}$ 81π.

16. The required depth h' is given by $wa^2h'^2(3h - h') = 3Wh^2$, w being the weight of unit volume of water

18. $\frac{(n+1)(2n+1)}{12n} g\varrho ah^2$, upper half, $\frac{(n+1)^2}{24n} g\varrho ah^2$, lower half, $\frac{(n+1)(3n+1)}{24n} g\varrho ah^2$, where a is the breadth and h the height of the vertical side

19. Draw lines at distances $\sqrt{1/n}$, $\sqrt{2/n}$, $\sqrt{3/n}$, . times the height of the rectangle, from the top

20. Draw radii OA_1, OB_1, OA_2, OB_2, , each pair equally inclined to the bounding diameter, at angles θ_1, θ_2, respectively, where $\sin\frac{\theta_1}{2} = \sqrt{\frac{1}{n}}$, $\sin\frac{\theta_2}{2} = \sqrt{\frac{2}{n}}$; etc.

22. There will be two positions of the tetrahedron $ABCD$, the edge AB being horizontal, in the first D is the highest vertex; and in the other, C. The tetrahedron can be brought from one position into the other by turning it about AB

23. Utilise the condition that the depth of the centre of gravity is constant

EXAMPLES 3

The depths of the c p are the following

2. $d + \frac{b}{3}\ \frac{2b+3d}{b+2d}$, where d is the depth of the upper side

3. Same as Q. 2, putting $d = h_1$ and $b = h_2 - h_1$.

4. $\dfrac{1}{2} \dfrac{a(3h_1^2 + 2h_1h_2 + h_2^2) + b(3h_2^2 + 2h_2h_1 + h_1^2)}{a(2h_1 + h_2) + b(2h_2 + h_1)}$.

5. $\dfrac{1}{2} \cdot \dfrac{d^4 - b^4}{d^3 - b^3}$.

11. The force $= \frac{1}{3}a^3 \times$ the weight of unit volume of water.

13. The distance of the line from the surface is $\dfrac{\sqrt{5} - 1}{2}$ times the vertical side

14. $4(5\sqrt{2} - 7)$ 3. The depth of water within the vessel is $\left(1 - \dfrac{1}{\sqrt{2}}\right)$ times an edge.

15. $\dfrac{1}{2} \cdot \dfrac{(\alpha + \beta)(\alpha^2 + \beta^2) \sim (\gamma + \delta)(\gamma^2 + \delta^2)}{(\alpha^2 + \alpha\beta + \beta^2) \sim (\gamma^2 + \gamma\delta + \delta^2)}$

EXAMPLES 4

1. $Y = \dfrac{1}{4} \cdot \dfrac{3\pi(a^2 + 4b^2) + 32ab}{4a + 3\pi b}$

2. $X = \dfrac{a}{2} \cdot \dfrac{3a + 8b}{4a + 3\pi b}$, Y as in Q 1 3 $Y = h + \dfrac{b^2}{4h}$.

4. $Y = h + \dfrac{a^2}{4h}\sin^2\theta + \dfrac{b^2}{4h}\cos^2\theta$.

5 $CP = \dfrac{b'}{4} \cdot \dfrac{3\pi b' + 16h'}{4b' + 3\pi h'}$, where C is the centre, b' is the semi-conjugate diameter and h' is the distance of the centre from the surface, along this line

6. $Y = \dfrac{2}{5} \dfrac{10h^2 - 15hk + 6k^2}{4h - 3k}$, where k is the distance of the bounding chord from the vertex

7. $Y = \dfrac{3a}{16\sqrt{2}}(\pi + 2)$.

8. $Y = \dfrac{3\pi}{16} \cdot \dfrac{a^4 - b^4}{a^3 - b^3}$, a and b being the radii

9. With reference to the asymptotes as axes,

$$X = \frac{c^2}{2(y_2 - y_1)} \log \frac{y_2}{y_1},$$

$Y = \frac{1}{2}(y_2 + y_1)$, where y_2, y_1 are the ordinates of the two horizontal lines.

11. $\frac{a}{24} \cdot \frac{64 - 15\pi}{4 - \pi}$.

13. Utilise the last result (in italics) of the triangle, Art 34, also, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

14. $\frac{23}{12\sqrt{3}} a$.

15. $\sin^{-1} \frac{2W}{W'}$, $\tan^{-1} \frac{3W}{2W'}$ and $\tan^{-1} \frac{3W}{W'}$, according as the box is tilted about one of the edges perpendicular to the edge just below and the edge parallel to the line of hinges, where W is the weight of the lid and W' that of the water contained in the box.

16. $\frac{n(3n+1)}{2n+1} \cdot \frac{b}{2}$; evaluate first the pressure on the portion in contact with the rth liquid and also its moment about the top Then find the sums.

17. (i) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{16h^2}$.

(ii) $X^2 + \left(Y - \frac{a^2 + b^2}{8h}\right)^2 = \left(\frac{a^2 - b^2}{8h}\right)^2$,

with reference to horizontal and vertical axes through the fixed centre

18. $X = \frac{a}{2}(\sin\theta + \cos\theta)$,

$$Y = \frac{a}{6} \cdot \frac{a(4 - 5\sin\theta\cos\theta) + 6b(\sin\theta - \cos\theta)}{2b + a(\sin\theta - \cos\theta)},$$

with reference to horizontal and vertical lines through A

EXAMPLES. 5.

2. $4 - \pi : 4 + \pi$. 3. $6b - \pi a \cdot 6b + \pi a$. 4. 6 cm.

5. $\frac{28\tan^2\alpha + 1}{16\tan^2\alpha + 1} W$ and $\frac{48\tan^3\alpha}{16\tan^2\alpha + 1} W$,

where W is the weight of the liquid contained.

6. $\tan\alpha = 2$.

EXAMPLES. 6.

1. $\frac{1}{8}wa^3\sqrt{8+\pi^2}$
2. The thrust on a lower part is three times that on an upper
3. $\frac{1}{3}wh^2\sqrt{a^2+b^2}$, where h is the height of the cone. 4. $\frac{2}{3}wa^3$.
5. $\frac{\sqrt{19}}{6}\pi wr^2h$, r being the radius of the base and h the height; the two thrusts are equally inclined to the horizontal.
6. Vertical thrust on one half $=\frac{2}{3}g\varrho\pi abc$,

   horizontal thrust $=g\varrho\pi ab\left[\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}+\frac{H\varrho_0}{\varrho}\right]$.

   The tangent at the highest point is horizontal; the first term within the bracket gives the depth of the centre below this tangent.
7. Vertical thrust $=\frac{2}{3}\pi a^3w$, w denoting the weight of unit volume of water Horizontal thrust

   $$=\frac{2}{3}wa^3-\frac{2}{3}wa^2y_1\cos^{-1}\frac{y_1}{a}+\frac{2}{3}w\sqrt{a^2-y_1^2}\,.\,(2a^2+y_1^2),$$

   where y_1 (depth of oil) satisfies the equation

   $$y^3-3a^2y+a^3=0.$$
8. Find horizontal component perpendicular to one of the given planes, and another component at right angles to the former
10. The fluid thrust on the whole solid is given by Art. 40, this is the resultant of the fluid pressures on the plane face and on the curved surface The former is known by the rules of Chapter III; hence the latter can be easily determined
11. $\frac{1}{8}wr^2\sqrt{16r^2+\pi^2h^2}$, acting in a direction making $\tan^{-1}\left(\frac{4r}{\pi h}\right)$ with the upward vertical
13. Inclination is $\tan^{-1}(3\sqrt{3})$ to the upward vertical

EXAMPLES. 7.

1. $\frac{w_0-w_2}{w_0-w_1}$
2. $g\frac{27Vg\varrho_0-29w}{2Vg\varrho_0+29w}$, ϱ_0 being the density of air at the earth's surface
3. $\frac{1}{20}\varrho_0+\frac{w}{gV}$ 6. $\sqrt[3]{\frac{2W}{\sqrt{3}}}$
7. GH, which is parallel to the base in this case, must be vertical.

8. Increased. (See Ex 3, Art 54.)

9. Increased by the weight of the volume $(v - V)$ of water.

10. $\frac{31}{40}$.

11. $6\sqrt[3]{5}$ inches.

13. GH will be parallel to the other side (or diagonal)

15. $5 \cdot \sqrt[3]{7.5}$ inches.

16. $1 : 2$, by volume.

17. The cylinder rises after water is added.

18. $\sqrt[3]{1 - \frac{1}{n^3}}$ of the axis.

19 Rises through a quarter of its length.

22. Less by $\frac{1}{99870}$th part of the original volume.

24. $\dfrac{1 - \frac{s}{\sigma}}{1 - \frac{s}{\varrho}} \cdot w$.

28. $\dfrac{8 - 5\sqrt{2}}{11}$.

30. $\tan^{-1} \dfrac{8cw}{3aW}$

31. $\frac{71}{27}$.

33. $\dfrac{dx}{dt}$ will be the velocity with which the candle burns if x denotes its length

34. The rods are not at right angles, the angle between them is acute. Let D, E, F be the middle points of AB, BC, BD. Then D would lie on the surface of water and the vertical line through D would divide (at right angles) FE in the ratio $2 : 1$

35. Tension $= \frac{1}{8}$th of the weight of each rod.

36. GH is perpendicular to FK where F, K are the middle points of the portions immersed, and

$$FH \cdot HK = BK \cdot BF.$$

Therefore $$FG^2 - KG^2 = FH^2 - HK^2,$$
whence the result can be deduced.

37 Let H' be the c g of the portions above the liquid and H that of the portions immersed. Then GHH' is a straight line; so that GH' is perpendicular to FK where F, K are the middle points of the portions above the surface. Utilise some result of the last question and the trigonometrical formula $(m+n) \cot \theta = m \cot \alpha - n \cot \beta$.

40. $(\frac{3}{7})^{\frac{3}{2}}$. Utilise the results (1) that the c g of a segment of parabola lies on the diameter BV bisecting the bounding chord PQ at V, and divides BV in the ratio $3:2$; (2) that the area of the segment $=\frac{4}{3}\sin\theta\ BV \,.\, VQ$ where θ is the angle between PQ and the diameter.

41. The first is symmetrical with one diagonal vertical and a portion of it (equal in length to half of a side) immersed. The other two are unsymmetrical, with an adjacent corner on the surface.

42. Work done by gravity $=2W\left(1-\frac{1}{n}\right)h$, where W denotes the weight of the cone Work done against the fluid thrust can be found, by calculus, to be

$$\int\frac{n^3x^3}{h^3}W\,dx \text{ or } \frac{n^3Wh}{4}\left(1-\frac{1}{n^4}\right).$$

Equating and simplifying, the result follows.

EXAMPLES. 8.

2. Parts of concentric spheres

3. Let EF be the line of floatation when AB is vertical, all lines of floatation pass through the middle point K of EF Next, take the line through K parallel to AB as the y-axis and the origin the c g. of the area immersed in this position The c g of the area immersed in any other position will be found to be given by $x=c_1\tan\theta$ and $y=c_2\tan^2\theta$, where c_1, c_2 are some constants and θ the angle between the line of floatation and the x-axis. Eliminate θ for the curve of buoyancy.

4. Area cut off by the chord of floatation $=$constant Let (x', y') be the coordinates of the middle point of the chord, and $y^2=4ax$ be the parabola, the above area would be given by $\frac{(4ax'-y'^2)^{\frac{3}{2}}}{3a}$ Since this is constant, $4ax'-y'^2=$constant, which gives the equation of the locus of (x', y'), or the curve of floatation For the position of H, see Q 40 of the last series

EXAMPLES. 9.

1. $HM=\frac{a^2\varrho}{4h\sigma}$, stable if $\frac{a^2}{2h^2}>\frac{\sigma}{\varrho}\left(1-\frac{\sigma}{\varrho}\right)$

2. $HM=\frac{h^2}{3\pi a}$, stable if $h>2a$

3. (i) $HM = \frac{3b^2}{4h}\sqrt[3]{\sigma}$; stable if $\frac{b^2}{h^2} + 1 > \frac{1}{\sqrt[3]{\sigma}}$ (ii) Replace b by a in the last results (iii) $HM = \frac{3\sqrt[3]{\sigma}}{4h}(b^2\cos^2\theta + a^2\sin^2\theta)$; stable if $b^2\cos^2\theta + a^2\sin^2\theta > \left(\frac{1}{\sqrt[3]{\sigma}} - 1\right)h^2$.

4. (i) $HM = \frac{4a^2}{3\pi b}$, stable. (ii) $HM = \frac{h^2}{3\pi b}$; stable if $h > 2b$

5. $HM = \frac{3}{4}\frac{(a+c)^2}{2a+c}$, neutral.

6. Find the maximum value of $\frac{\sigma}{\varrho}\left(1 - \frac{\sigma}{\varrho}\right)$.

7. $HM = 2a$; $HG = \frac{2}{3}(h - h')$

8. (i) $HM = \frac{a\varrho}{12\sigma}$ where ϱ is the sp gr. of the liquid; stable if $\frac{\sigma}{\varrho} < \frac{3-\sqrt{3}}{6}$, also if $\frac{\sigma}{\varrho} > \frac{3+\sqrt{3}}{6}$ but less than unity (ii) *If* $\sigma < \frac{1}{2}\varrho$, $HM = \frac{2a}{3}\sqrt{\frac{\sigma}{\varrho}}$, stable if $\frac{\sigma}{\varrho} > \frac{9}{32}$ Secondly, if $\sigma > \frac{1}{2}\varrho$ *but* $< \varrho$, stable if $\frac{\sigma}{\varrho} < \frac{23}{32}$. $HM = \frac{2a\varrho}{3\sigma}\left(1 - \frac{\sigma}{\varrho}\right)^{\frac{3}{2}}$.

9. $\frac{Wa\theta}{Wa + wh}$.

12. See Ex. 2, Art 70, only the symmetrical position possible Since the sp gr $> \frac{9}{16}$, the plane of floatation is a rectangle whose breadth is $> \frac{3}{4}a$ and length $= h$, where a is the length of a side of the base and h is the length of the prism. Thus, the maximum value of $x(a - x)$ occurs when x is least, *i e* $x = \frac{3}{4}a$.

16. Vertical angle should not exceed 90°.

17. Stable if the c g. of the buoy lies below the base of the hemispherical portion.

EXAMPLES. 10.

3. $\frac{dp}{dr} = -\frac{\lambda\varrho}{r}$; integrate, put $r = a - x$ and expand in power series.

5. Let C be the fixed point, attraction at P is $\lambda . PC$; let CD be drawn parallel to the constant force such that $\lambda . CD =$ the force. Then the resultant force at P is towards D.

EXAMPLES. 11.

1. Divide the base into thin concentric rings and apply the method of calculus

2. $\dfrac{2\sqrt{gh}}{a}$, where h is the height and a the radius

3. The vertex of the cone is on the free surface Pressure $=3\left(1+\dfrac{\omega^2 h\tan^2\alpha}{4g}\right)$ times the weight of the liquid

4. The vertex of the free surface should be below the base of the vessel (See Ex 1, Art 85) Angular velocity $=\dfrac{2\sqrt{gh}}{a}$.

7. $\dfrac{\pi\omega^2 a^3}{2g}$. (See Ex 6, Art. 85.)

8. $(\sqrt{3}+1)\sqrt{\dfrac{2g}{3a}}$ Obtain two expressions for the vertical distance between the extremities of the liquid, and equate

10. $\sqrt{\dfrac{2g}{h}}$

11 If $\omega =$ or $> \sqrt{\dfrac{2g}{l}}$, where l is the latus rectum of the generating parabola of the vessel, no liquid can remain in the vessel.

12 Since the vertex is always below any possible free surface, the pressure there is greatest and greater than the atmospheric pressure

15 A length $AD = a\sqrt{1-\dfrac{2bg}{a^2\omega^2}}$ is empty

16. Pressure at A is least, so it should not be negative.

17 $\frac{1}{2}a\rho\,[a\omega^2\sin^2\theta+2g(1-\cos\theta)]$, where θ is the angular distance from the highest point, the pressure is maximum where $\cos\theta = -\dfrac{g}{a\omega^2}$, a quantity which should be less than unity in magnitude.

18. At the point where the pressure is greatest (see the previous question)

19. No liquid will escape if the lowest point be on or above the free surface.

22 $\sqrt{\frac{7g}{6a}\cot\alpha}$

24 The point is g/ω^2 above the vertex of the base

25. If the vertical line be not the tangent, p would denote the length of the perpendicular on this line from the centre

27. Either use double integrations, or find the pressure and its moment about Ox, on a narrow horizontal strip and then integrate these for the whole rectangle.

EXAMPLES. 12.

1. $26\frac{98}{127}$ ft 2. 841631 nearly. 3. $3 . \sqrt[3]{17}$. 4. 121 5 c c

6. The distances from the lower end of the tube are 1 7 cm. and 30 8 cm nearly.

7. $\frac{1}{2}h + 7H$.

8. The fall is the positive root of the equation

$$x^2 + ax - \frac{273+t}{273}Hb = 0$$

9. Use the equation (23) 11. 7 in

13. $26\frac{1}{3}$ in, $(30 - \sqrt{6})$ in

14. Neglect terms of the second degree in h and the lengths required, since they are small; $\frac{h(H-c)}{a+2H-2c}$

15. The level of mercury in the tube will remain at the same height above the level in the cistern; a volume of mercury, whose weight is equal to that of the piece of glass, would flow out from the tube into the cistern The effect of the capillary action is not considered

17. $\frac{h+a\sigma}{3h+a\sigma}\ a$ 19. $\frac{4}{3}a$

20. The condition for equilibrium in a floating position is $W \leqslant V'w < Vw$, where V' is the volume of the compressed air in the vessel (or the volume of the water displaced) when it is just completely immersed and w the weight of unit volume of water If $W > V'w$, the vessel would sink, because its weight exceeds the fluid pressure on it (See Art 40.) In the former case, there are two positions of equilibrium, viz (1) when the vessel is partially immersed, and (2) when it is com

pletely immersed (at great depth), the position in each case being given by the condition $W = V''w$ where V'' is the volume of the water displaced [See Ex 2, Art 133]

22. The cone is in equilibrium under its weight, the pressure of the enclosed gas, the atmospheric pressure and the reaction of the vessel acting at points along the circle of contact, resolve vertically

24. $\dfrac{a\alpha(b+\beta)}{ab+\alpha\beta}$ and $\dfrac{b\beta(a+\alpha)}{ab+\alpha\beta}$ **25.** $\frac{1}{8}Wd^2$.

27. Consider the equilibrium of the barometer tube, the vertical forces on it are (1) its weight, (2) the tension of the string, both of which are constant, (3) upward fluid thrust due to mercury (at the base), and (4) downward pressure due to atmosphere (on the top) The horizontal pressures need not be taken into consideration

EXAMPLES. 13.

2. 9 9 miles nearly.

5 $p'\left(\dfrac{p}{p'}\right)^{z/h}$, if p' be the pressure at the lowest point

7. 4850 ft **9** $\dfrac{pv^\gamma}{1-\gamma}[v_1^{1-\gamma} - v_2^{1-\gamma}]$.

10. Take the pressure at the free surface of the atmosphere to be zero

11 Take $\dfrac{p}{\varrho} = kT$, and $\dfrac{dT}{dz} = -\lambda$ It can be shown that for the value $\dfrac{g}{k}$ of λ, $\dfrac{d\varrho}{dz} = 0$

12. The constants may involve the height of and the temperature at the lower points which are regarded as fixed

13. Since the diameter of the cross-section of the tube is small, p is constant in the same vertical line within the tube, also, temperature is assumed to be constant, ∴ $p = k\varrho$ See Q 1, and the equation (2) of Art 72, also the method of treating the rotating liquids.

14. Find the velocity of the piston generated by the impulse, then apply the principle of energy and work See the result (33) of Ex 2, Art 112.

EXAMPLES. 14.

2. Gold, 00028 cu. ft , silver, 00062 cu ft 3 $\frac{2}{5}$.

4. Volumes are as 35 37 , the masses are as 490 481

5. Greater, since it displaces a greater volume of air

7. Gold, $31\frac{1}{72}$ gr , diamonds, $7\frac{35}{72}$ gr , rubies, 6 gr

EXAMPLES. 15.

1. 5 grammes 2. 0 8

3. $\varrho(1-\sigma)+\sigma$. Take v as the volume and s the sp gr of the material of which the bottle is made, and V its capacity (See Ex 2, Art 118)

4. $\dfrac{b-c\sigma}{a}$

5. The fluid pressure at any level depends on the height of the water column, and not on the presence of solids

EXAMPLES. 16.

1. 10 cm. 2. $\frac{147}{163}$. 3. 62 grammes 4. 99 . 106

6. Graduations will be at distances $\frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}$, from O (Fig. 69), where $OA=a$.

7. Sp. gr $=\dfrac{w-w_1}{w_2-w_3+w-w_1}$, where w, w_1, w_2, w_3 are the weights placed on the upper pan along with (1) nothing else, (2) the solid on the upper pan, (3) the solid and the sinker in the lower cup, and (4) the sinker in the lower cup

8. The ratio is $\dfrac{W-P}{W}$, where W is the weight of the whole instrument and P that of the part removed.

9. $1-\dfrac{n}{4}$.

10. See Ex 1, Art 129. Assume, moreover, that V is the volume of the solid.

12. $s=\dfrac{w-w_1}{w_2-w_1}\cdot s_2+\dfrac{w-w_2}{w_1-w_2}s_1$.

13. Area of the cross-section $=\dfrac{c}{(a+bx)^2}$, where x is the distance of the section from the top and a, b, c are some constants Take x, $x+dx$, $x+2dx$, , and s, $s+ds$, $s+2ds$, . as corresponding values of x and the sp. gr., so that the differential coefficient $\dfrac{ds}{dx}$ would be a constant

14. $1\frac{157}{384}$. Assume V_1 and s to be the volume and the sp. gr. of the weight attached, also V to be the volume of the water used to be displaced by the hydrometer. Two equations can be obtained, by the principle of Archimedes, for each liquid; whence one equation involving $\dfrac{V_1}{V}$, s and the sp. gr. of the liquid can be easily obtained for each liquid.

EXAMPLES. 17.

2. The pressure of the (enclosed) air is greater than the water pressure at the hole

3. The following fractions of the volume of the bell:

$$\text{(i)}\ \frac{4h-3d}{9h},\quad \text{(ii)}\ \frac{31h-21d}{144h}.$$

4. 300 cu. in.

7. $(h'-h)\left(\varrho+\dfrac{bH}{hh'}\right)$

8. 46 83 ft nearly

9 $\dfrac{2-5\varrho}{3-6\varrho}$

11. 13.

12. The sp gr of sea-water, viz $\sigma > 1$. Let x, x' be the lengths of air column in the two cases. Thence

$x^2+(h+a)x = x'^2\sigma+(h+a\sigma)x' < x'^2\sigma^2+(h+a)x'\sigma'$.

Therefore $x < x'\sigma$ Use (5) of Art. 133.

EXAMPLES. 18.

1. 22 58 ft. and 33 20 ft nearly

2. 25 98 ft. nearly.

3. Show that x_2, as obtained from (7) and the next equation of Art. 136, is greater than a.

4. 3 · 1.

5. 15.

6. When the distance of the piston from the top of the barrel is $\left(\frac{V}{V+v}\right)^n$ times its length.

7. $\frac{1}{8}$ of the density of the atmospheric air.

11. $\left[\frac{n(a+b+c)+b}{n(a+b+c)+b+c}\right]^2$ times the initial pressure.

12 Pressure at the topmost point of the siphon due to the arm dipping in the vessel must exceed that due to the other arm in order that the action of the siphon may continue; the pressure of the air within the vessel diminishes as water flows out.

14. See Ex 2, Art 112

15. See Exs. 2 and 3, Art 150, prove that $p < \frac{a+b}{b} P$

16. Assume ϱ_1, ϱ_2 to be the densities after 1, 2, strokes; ϱ_n will be equal to $\frac{C}{A+B} \varrho + \frac{A}{A+B} \cdot \varrho_{n-1} =$.

17. Use a method similar to Q 16 Show that $\mu < 1$, therefore, when n is infinitely large, $\mu^n = 0$, and the limiting density $= \frac{CC'}{(B-C)(B-C')} \varrho$ (Cf Ex 2, Art 150)

18 See Q 6 and Art 141

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